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THE FARGUES–FONTAINE CURVE AND DIAMONDS [d'après Fargues, Fontaine, and Scholze]

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Contents

1. INTRODUCTION

The goal of this text is to overview the Fargues–Fontaine curve, its role in p-adic Hodge theory, and its relation to Scholze's theory of perfectoid spaces and diamonds. On the other hand, we do not touch on the role of the curve in local class field theory [11, 14] or in the local Langlands correspondence [13].

1.0.1. *The literature.* — The definitive text on the foundations of the curve is the book by Fargues and Fontaine [16]. There exist several more introductory articles, in particular Colmez's extensive preface [7] to the book, Fargues' recent ICM text [15], and Fargues–Fontaine's Durham survey [17]. In view of these articles, which were very useful when preparing the current text and which we highly recommend to readers with a background in p-adic arithmetic geometry, we have attempted to present the theory here with the non-expert in mind. In particular, Sections 2–3 should be accessible to any reader with a knowledge of elementary algebraic geometry.

Concerning diamonds, Scholze's Berkeley lecture notes [32] contain the main concepts, while [31] is the source for the technical foundations, and his ICM text [29] gives an overview. Section 4 on diamonds is sparse on details but we have attempted to indicate some of the main ideas of the theory.

1.0.2. *What is the curve?* — Let us begin by recalling the old analogy between the integers $\mathbb{Z}$ and the ring $\mathbb{C}[z]$ of polynomials in one variable over the complex numbers. They are both principal ideal domains, even Euclidean domains, with Euclidean function given respectively by the usual absolute value $|\cdot|$ coming from $\mathbb{R}$ and by polynomial degree. Geometrically, $\mathbb{C}[z]$ (whose monic prime polynomials identify with the complex plane via $x \mapsto z - x$) is the set of functions on the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ whose only pole is at infinity, and the degree of a polynomial is precisely the order of this pole. Analogously, arithmetic geometry views $\mathbb{Z}$ as functions on the set of prime numbers, with an extra point at infinity being provided by the real numbers or equivalently by $|\cdot|$. Motivated by this analogy, it is not uncommon to develop analogues of geometric tools for the Riemann sphere (e.g., vector bundles, sheaves, cohomology,...) when doing arithmetic geometry over $\mathbb{Z}$. This approach, although fruitful, can only be taken so far, since the point at infinity for $\mathbb{Z}$ is no longer algebraic and so the compactification-at-infinity $\{\text{primes}\} \cup \{|\cdot|\}$ is no longer an algebro-geometric object.

The theory of Fargues and Fontaine takes this analogy much further if we focus on a given prime number p and replace arithmetic geometry over $\mathbb{Z}$ by arithmetic geometry over $\mathbb{Z}_p$ or $\mathbb{Q}_p$. The Euclidean domain $\mathbb{Z}$ or $\mathbb{C}[z]$ is now replaced by a certain $\mathbb{Q}_p$-algebra B_e (coming from p-adic Hodge theory), which is again (almost) a Euclidean domain with the Euclidean structure arising from a point at infinity. But, whereas in the case of $\mathbb{Z}$ the point at infinity was outside the world of algebraic geometry, we are now in a situation much closer to that of the Riemann sphere: there exists an actual curve (in a sense of algebraic geometry) X^{FF} whose functions regular away from a certain point at infinity are the ring B_e and whose geometric and cohomological properties (which are similar to those of $\mathbb{P}^1_{\mathbb{C}}$) encode significant information about arithmetic geometry over $\mathbb{Q}_p$. This is the *fundamental curve of p-adic Hodge theory*, or the *Fargues–Fontaine curve.*

1.0.3. *Overview.* — We will return to the above point-at-infinity perspective after Theorem 1.1 but first, given our goal of diamonds, we wish to introduce the Fargues–Fontaine curve as a space of untilts. Here "untilt" refers to the tilting–untilting correspondence of Scholze through which one passes between geometry over the characteristic zero field $\mathbb{Q}_p$ and over the characteristic p field $\mathbb{F}_p$ [23, 30]. For example, let $\mathbb{C}_p$ be the "p-adic complex numbers", i.e., the p-adic completion of the algebraic closure of $\mathbb{Q}_p$; then its *tilt* $\mathbb{C}_p^\flat$, whose definition we will recall in Section 2.1, is a field with similar superficial structure to $\mathbb{C}_p$ but it is an extension of $\mathbb{F}_p$ rather than of $\mathbb{Q}_p$. A fundamental motivating question for both the curve and for diamonds is the following:

> Putting $F = \mathbb{C}_p^\flat$, do there exist fields $C \supseteq \mathbb{Q}_p$ other than $\mathbb{C}_p$ such that $C^\flat = F$? Informally, do there exist other ways of passing back to characteristic zero from characteristic p ?

More precisely, since equality is clearly not the right notion, let $|Y_F|$ denote the set of *untilts* (C, ι), where C is a suitable extension of $\mathbb{Q}_p$ and $\iota : F \xrightarrow{\simeq} C^\flat$ is a specified isomorphism; such pairs are taken up to an obvious notion of equivalence. A coarser notion of equivalence is obtained by taking the Frobenius automorphism $\varphi : F \xrightarrow{\simeq} F$, $x \mapsto x^p$, into account, thereby leading to the set of untilts up to Frobenius equivalence $|Y_F|/\varphi^{\mathbb{Z}}$. The remarkable theorem of Fargues and Fontaine states that this set of untilts admits the structure of a "smooth, complete curve" (see Definition 2.6), now known as the *Fargues–Fontaine curve* X_F^{FF} *of* F.

Theorem 1.1 (Fargues–Fontaine). — *The set $|Y_F|/\varphi^{\mathbb{Z}}$ is the underlying set of points of a complete curve X_F^{FF}.*

We can now make more precise paragraph 1.0.2 about points at infinity; see Sections 2.2–2.3 for details. The original field $\mathbb{C}_p$ is itself an untilt of F, thereby giving us a preferred point $\infty \in X_F^{\mathrm{FF}}$. The ring of functions on X_F^{FF} which are regular away from ∞ turns out to equal the Frobenius-fixed subring $B_e := B_{\mathrm{crys}}^{\varphi=1}$ of the classical crystalline period ring of Fontaine [21]; meanwhile, the completed germs of meromorphic functions at ∞ equals his classical de Rham period ring B_{dR}. The classical (and subtle) so-called *fundamental exact sequence* of p-adic Hodge theory

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \longrightarrow 0$$

then translates into a simple cohomological vanishing statement about the curve X_F^{FF}. In this way the Fargues–Fontaine curve may be viewed as subtly gluing together B_e (which is almost a Euclidean domain) and B_{dR} (which is a complete discrete valuation field), in the same way as the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ glues together $\mathbb{C}[z]$ (= functions on $\mathbb{P}^1_{\mathbb{C}}$ regular away from infinity) and $\mathbb{C}((\frac{1}{z}))$ (= completed germs of meromorphic functions at infinity). Moreover, just as for $\mathbb{P}^1_{\mathbb{C}}$, the sum of the orders of zeros/poles of any meromorphic function on X_F^{FF} is zero, which is precisely what it means for X_F^{FF} to be "complete".

Another similarity between the Fargues–Fontaine curve and the Riemann sphere is their vector bundles. On the Riemann sphere, a theorem of Grothendieck states that any vector bundle is isomorphic to $\bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\lambda_i)$ for some unique sequence of integers $\lambda_1 \geq \cdots \geq \lambda_m$, where $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\lambda)$ is the usual twisted line bundle of degree λ. On the Fargues–Fontaine curve the situation is more complicated, as there exist non-decomposable "rational twists" $\mathcal{O}_{X_F^{\mathrm{FF}}}(\lambda)$, for $\lambda \in \mathbb{Q}$ (this is only a line bundle if $\lambda \in \mathbb{Z}$; in general its rank is given by the denominator of λ), but then Fargues and Fontaine establish the following analogue of Grothendieck's theorem:

Theorem 1.2 (Fargues–Fontaine). — *Let E be a vector bundle on X_F^{FF}. Then there exists a unique sequence of rational numbers $\lambda_1 \geq \cdots \geq \lambda_m$ such that E is isomorphic to $\bigoplus_{i=1}^m \mathcal{O}_{X_F^{\mathrm{FF}}}(\lambda_i)$.*

The proof of Theorem 1.2, which we discuss in Section 3.2 but which is beyond the scope of this survey, requires a range of deep techniques including p-divisible groups and p-adic period mappings. Conversely, it encodes enough information to have important applications to classical questions in p-adic Hodge theory. For example, we use it in Section 3.3 to explain a short proof of Fontaine's "weakly admissible implies admissible" conjecture about Galois representations from 1988 [22] (resolved first by Colmez–Fontaine in 2000 [8]). The key idea is that many linear algebraic objects of p-adic Hodge theory (modules with filtration, with Frobenius,...) may be used to build vector bundles on X_F^{FF}, which may then be analysed through Theorem 1.2. An important technique in such analyses is the general rank-degree formalism of Harder and Narasimhan [25], which applies to vector bundles on any curve; we review their theory in Section 3.1.

We now turn to Scholze's theory of diamonds. Recall that our motivating goal is to classify untilts of the characteristic p field F. In the world of diamonds, such an untilt corresponds to a "morphism" from $\mathbb{Q}_p$ to F: of course, algebraically there exist no homomorphisms between fields of different characteristic, but diamonds provide a theory of p-adic geometry in which everything is of characteristic p in some sense. Even more interestingly, the choice of two untilts of F (i.e., two points of $|Y_F|$) corresponds to a morphism from $\mathbb{Q}_p \otimes \mathbb{Q}_p$ to F, where $- \otimes -$ refers to an absolute tensor product for diamonds. (To avoid misleading the reader, we caution that there is no set-theoretic object $\mathbb{Q}_p \otimes \mathbb{Q}_p$, nor set-theoretic map $\mathbb{Q}_p \to F$, only the associated diamond.) Weil's simple proof [34] of the Riemann hypothesis for a curve $\mathcal{C}$ over a finite field $\mathbb{F}_q$ crucially depends on the geometry of the surface $\mathcal{C} \times_{\mathbb{F}_q} \mathcal{C}$, and a well-known philosophy predicts that there should exist a similar object "$\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$" in arithmetic geometry. Diamonds appear to provide this object p-adically. (We emphasise that this is not an empty philosophy: the "shtukas" of Drinfeld [9] which are central in the geometric Langlands correspondence for $\mathcal{C}$ also involve $\mathcal{C} \times_{\mathbb{F}_q} \mathcal{C}$, and Fargues and Scholze's ongoing work on arithmetic local Langlands uses diamonds to develop an analogous theory over $\mathbb{Q}_p$ [18, 32].)

In Section 4 we attempt to explain these ideas more precisely by defining the category of diamonds. Scholze associates to any reasonable adic space X (e.g., the analytification of a variety over a non-archimedean field such as $\mathbb{Q}_p$, $\mathbb{C}_p$, or F) a diamond which classifies certain untilts of perfectoid spaces. For example, $\mathbb{Q}_p$ and F themselves give rise to diamonds $\mathrm{Spd}(\mathbb{Q}_p)$ and $\mathrm{Spd}(F)$ and, as we suggested in the previous paragraph, morphisms of diamonds $\mathrm{Spd}(F) \to \mathrm{Spd}(\mathbb{Q}_p)$ are exactly the untilts of F (the morphism has changed direction, as usual when passing from algebra to geometry). From the point of view of diamonds, the Fargues–Fontaine curve gains the following beautiful interpretation:

Theorem 1.3 (Scholze). — *The diamond associated to the Fargues–Fontaine curve X_F^{FF} is naturally isomorphic to the product*

$$\mathrm{Spd}(F)/\varphi^{\mathbb{Z}} \times \mathrm{Spd}(\mathbb{Q}_p).$$

Finally, in Section 5 we give a detailed sketch of the construction of the Fargues–Fontaine curve; this is necessarily slightly technical (though, in principal, it only requires some elementary algebraic geometry and some comfort manipulating large p-adic algebras) and may be safely ignored by readers uninterested in the actual construction. It begins by observing that Fontaine's infinitesimal period ring $A_{\mathrm{inf},F} := W(\mathcal{O}_F)$ (i.e., Witt vectors of the ring of integers of F) may be naturally viewed as a ring of functions on the set $|Y_F|$. Fargues and Fontaine substantially develop this point of view by introducing a topological structure on $|Y_F|$ and replacing $A_{\mathrm{inf},F}$ by a larger ring of functions B_F; this is the largest reasonable ring of continuous functions on $|Y_F|$ in the sense that $y \mapsto \{f \in B_F : f(y) = 0\}$ identifies $|Y_F|$ with the closed maximal ideals of B_F (see Prop. 5.4). Moreover, each of these ideals is principal, generated by a so-called primitive element of degree one, indicating that $|Y_F|$ is one-dimensional in some sense.

The Frobenius action on $|Y_F|$ from before Theorem 1.1 turns out to be properly discontinuous, whence $|Y_F|/\varphi^{\mathbb{Z}}$ inherits a topology making it locally homeomorphic to $|Y_F|$. The next step is to construct functions on $|Y_F|/\varphi^{\mathbb{Z}}$. Unfortunately, the only φ-invariant functions on $|Y_F|$ are constant. Instead, Fargues and Fontaine develop a theory of Weierstrass products to construct, for each point $y \in |Y_F|$, a function $t_y \in B_F$ satisfying $\varphi(t_y) = pt_y$ and with a simple zero at each point of the discrete set $\varphi^{\mathbb{Z}}(y) \subseteq |Y_F|$ and no other zeros or poles. So, given any other function $g \in B_F$ satisfying $\varphi(g) = pg$, we obtain a meromorphic function g/t_y on $|Y_F|/\varphi^{\mathbb{Z}}$ which is regular away from the image of y. Fargues and Fontaine prove that this process generates all functions on $|Y_F|/\varphi^{\mathbb{Z}}$ or rather, in more precise algebro-geometric language:

Theorem 1.4 (Fargues–Fontaine). — *(1) The graded ring $\bigoplus_{k\geq 0} B_F^{\varphi=p^k}$ is graded factorial, with irreducible elements of degree one.*
(2) The closed points of the scheme $\mathrm{Proj}(\bigoplus_{k\geq 0} B_F^{\varphi=p^k})$ canonically identify with the set $|Y_F|/\varphi^{\mathbb{Z}}$.

Part (1) of Theorem 1.4 is a central result in the entire theory; in particular, it more or less formally implies that the Fargues–Fontaine curve, which we may now define, really is a curve:

Definition 1.5. — The Fargues–Fontaine curve is

$$X_F^{\mathrm{FF}} := \mathrm{Proj}(\bigoplus_{k\geq 0} B_F^{\varphi=p^k}).$$

We will see earlier in Section 2.3 a similar (and ultimately equivalent) definition of X_F^{FF} in terms of the crystalline period ring.

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All results herein are due to Fargues, Fontaine, and Scholze, while all errors and excessive simplifications are due to the author.

2. EXISTENCE OF THE CURVE

In this section we present two points of view on the Fargues–Fontaine curve. The first of these, in §2.1, states that the curve classifies certain equivalence classes of untilts of a perfectoid field. The second, given in §2.2, instead characterises it via its functions (given by a certain period ring of p-adic Hodge theory) together with a point at infinity (corresponding to another period ring). In §2.3 we use the second approach to derive and motivate a precise scheme-theoretic definition of the curve.

2.1. As a space of untilts of a perfectoid field

The first interpretation of the Fargues–Fontaine curve which we will explore, which is the most important from the point of view of Scholze's diamonds, is that it classifies untilts of perfectoid fields. We therefore begin by summarising the essential ideas of tilting and untilting.

Let C be a field with the following properties:

(Pf_0) C is algebraically closed, contains $\mathbb{Q}_p$, and is complete with respect to a non-archimedean absolute value $|\cdot|_C : C \to \mathbb{R}_{\geq 0}$ extending the usual p-adic absolute value on $\mathbb{Q}_p$.

For example, the standard choice is to take $C = \mathbb{C}_p$ the "p-adic complex numbers", i.e., the completion of the algebraic closure of $\mathbb{Q}_p$. The *tilt* $F = C^\flat$ of C (to employ Scholze's terminology [30]; the construction exists since the early days of p-adic Hodge theory) will be a field with the following similar properties but now of characteristic p:

(Pf_p) F is algebraically closed, contains $\mathbb{F}_p$, and is complete with respect to a non-trivial non-archimedean absolute value $|\cdot|_F : F \to \mathbb{R}_{\geq 0}$.

There are other ways in which C and its tilt $C^\flat$ are related: they have the same residue field and there is a multiplicative map $\# : C^\flat \to C$, called the *untilting map*.

Let us now define the tilt $C^\flat$. As a set, $C^\flat$ is the set of p-power compatible sequences in C, i.e.,

$$C^\flat := \{(a_0, a_1, \dots) : a_i \in C,\ a_i^p = a_{i-1}\}. \tag{1}$$

We define multiplication in $C^\flat$ term-wise, while addition is defined by the rule

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) := (c_0, c_1, \dots), \qquad \text{where } c_i := \lim_{i \le n \to \infty} (a_n + b_n)^{p^{n-i}}, \tag{2}$$

where the limit is a convergent limit in the complete field C. It is not hard to check, and in any case classical, that these operations give $C^\flat$ the structure of a perfect field of characteristic p. The promised untilting map $\# : C^\flat \to C$ is simply the canonical projection $\alpha = (a_0, a_1, \dots) \mapsto \alpha^\# := a_0$, and then the absolute value $|\cdot|_C$ on C induces an absolute value on $C^\flat$ by $|\alpha|_{C^\flat} := |\alpha^\#|_C$. It is easy to check that $C^\flat$ is complete under this absolute value and has the same residue field as C; careful use of Hensel's lemma shows that it is also algebraically closed. We refer the reader to [23, §1.3] [30] for more on tilting.

Example 2.1. — (1) Fixing a compatible sequence of p-power roots of p (respectively, of unity), we obtain elements

$$p^\flat := (p, p^{1/p}, p^{1/p^2}, \dots), \qquad \varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots)$$

of the field $C^\flat$ which play an important role.
(2) If $C = \mathbb{C}_p$, then $C^\flat$ is isomorphic to the completion of the algebraic closure of the Laurent series field $\mathbb{F}_p((t))$, in such a way that $t \mapsto p^\flat$.

Let us now attempt to go in the opposite direction: given a field F as in (Pf_p), does there exist a field C as in (Pf_0) such that $C^\flat = F$? Can we classify all such C ? To make this more precise, one defines an *untilt* of F to be a pair (C, ι) where C is a field with properties (Pf_0) and $\iota : F \xrightarrow{\simeq} C^\flat$ is an isomorphism of valued fields.(1) We say that two untilts (C, ι) and (C', ι') are *equivalent* if there exists an isomorphism $C \cong C'$ such that the induced isomorphism between their tilts is compatible with ι and ι'. Although it might not be clear at present, we will see using elementary algebra in Proposition 5.1 that F admits many untilts; in particular the following set $|Y_F|$ is non-empty.

Definition 2.2. — Let $|Y_F|$ be the set of equivalence classes of untilts of F.

Given an untilt (C, ι) of F, we may construct new untilts $(C, \iota \circ \varphi^m)$, for all $m \in \mathbb{Z}$, where $\varphi : F \xrightarrow{\simeq} F$, $x \mapsto x^p$ is the absolute Frobenius automorphism. These are not interesting new untilts: therefore we say that untilts (C, ι) and (C', ι') are *Frobenius*

(1) More precisely this would usually be called an "untilt of characteristic zero" in the literature.

equivalent if there exists $m \in \mathbb{Z}$ such that (C, ι) and $(C', \iota' \circ \varphi^m)$ are equivalent. The set of Frobenius equivalence classes of untilts is given by the quotient

$$|Y_F|/\varphi^{\mathbb{Z}}$$

where the infinite cyclic group $\varphi^{\mathbb{Z}}$ is acting on $|Y_F|$ via $\varphi^m(C, \iota) := (C, \iota \circ \varphi^m)$.

The first existence statement of the Fargues–Fontaine curve asserts that this space of untilts $|Y_F|/\varphi^{\mathbb{Z}}$ admits the structure of a "smooth, complete curve", and that it therefore makes sense to study its geometric and cohomological properties. We refer to Definition 2.6 for the precise definition of a complete curve.

Theorem 2.3 (Fargues–Fontaine). — *There exists a complete curve X_F^{FF} whose points*(2) *are in natural bijection with $|Y_F|/\varphi^{\mathbb{Z}}$. (Moreover, the point of X_F^{FF} corresponding to a given untilt (C, ι) has residue field C.)*

To simplify notation, we fix a field F as in (Pf_p) *for the rest of the text, and write*

$$|Y| = |Y_F|, \qquad X^{\mathrm{FF}} = X_F^{\mathrm{FF}}.$$

The reader is welcome to suppose that F is the tilt of $\mathbb{C}_p$.

2.2. As a period ring with a point at infinity

The next point of view on the Fargues–Fontaine curve is that it is obtained by compactifying (the spectrum of) a period ring of p-adic Hodge theory by adding a point at infinity, in the same way as the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ is obtained from the complex plane by adding a point at infinity.

We begin with some reminders on Riemann surfaces(3) and their functions. Let X be a Riemann surface (or more generally a smooth projective curve over any algebraically closed field) and let $\mathbb{C}(X)$ denote its field of meromorphic functions. Given a meromorphic function $f \in \mathbb{C}(X)$, its order of vanishing $\mathrm{ord}_x(f)$ at a point $x \in X$ is defined as usual: namely, expanding the function as $f = \sum_{n \gg -\infty} a_n z_x^n$, where z_x is a local coordinate at x, we have

$$\mathrm{ord}_x(f) := \min\{n : a_n \neq 0\} \in \mathbb{Z} \cup \{\infty\}.$$

Each function $\mathrm{ord}_x : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$ is a valuation,(4) and we recall the classical *degree formula* that

$$\sum_{x \in X} \mathrm{ord}_x(f) = 0 \tag{3}$$

for all $f \in \mathbb{C}(X)$, which reflects the compactness of X.

The Fargues–Fontaine curve most closely parallels the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, so let us now consider that case from a more algebraic point of view. Let $\infty \in \mathbb{P}^1_{\mathbb{C}}$ be the point at

(2) *Points* mean closed points.

(3) Always compact and connected.

(4) i.e., ord_x is surjective, multiplicative, and satisfies $\mathrm{ord}_x(f) = \infty \Leftrightarrow f = 0$ and the non-archimedean inequality $\mathrm{ord}_x(f + g) \leq \max\{\mathrm{ord}_x(f), \mathrm{ord}_x(g)\}$.

infinity, so that $\mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$ identifies with the complex plane. The field of meromorphic functions on $\mathbb{P}^1_{\mathbb{C}}$ is precisely $\mathbb{C}(z)$, i.e., the field of rational functions in one variable z; meanwhile, the meromorphic functions which are regular away from infinity form the algebra $\mathbb{C}[z] \subseteq \mathbb{C}(z)$ of polynomials in z.

The theory of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ is almost completely encoded in the algebra $\mathbb{C}[z]$, except that we have lost the point at infinity; to record this extra point, we keep track of the valuation ord_∞. In conclusion, we can completely encode the Riemann sphere in the following algebraic pair of data:

$$\text{algebra } \mathbb{C}[z], \text{ valuation on its field of fractions } \mathrm{ord}_\infty : \mathbb{C}(z) \to \mathbb{Z} \cup \{\infty\}.$$

The degree formula (3) now takes the algebraic form

$$\mathrm{ord}_\infty(f) + \sum_{\mathfrak{p} \subseteq \mathbb{C}[z]} \mathrm{ord}_{\mathfrak{p}}(f) = 0 \tag{4}$$

for all $f \in \mathbb{C}[z]$, where $\mathfrak{p}$ runs over the non-zero prime ideals of $\mathbb{C}[z]$ (each such ideal $\mathfrak{p}$ is generated by $z - x$ for some unique $x \in \mathbb{C}$, and then ord_x is precisely the associated $\mathfrak{p}$-adic valuation $\mathrm{ord}_{\mathfrak{p}}$ on $\mathbb{C}[z]$). The Fargues–Fontaine curve will provide us with another example of such a pair.

We first further analyse the properties of the pair $(\mathbb{C}[z], \mathrm{ord}_\infty)$. Given $f \in \mathbb{C}[z]$, its order of vanishing $\mathrm{ord}_\infty(f)$ at the point ∞ is simply $-\deg(f)$: indeed, the expansion of $f = \sum_{n=0}^{\deg f} a_n z^n$ in terms of the coordinate $z_\infty = 1/z$ at infinity is exactly $f = \sum_{n=-\deg f}^{0} a_n z_\infty^n$. Thus $-\mathrm{ord}_\infty = \deg : \mathbb{C}[z] \to \mathbb{N} \cup \{-\infty\}$ is the usual degree function, which is a Euclidean function on $\mathbb{C}[z]$. Recall here that a *Euclidean function* on an integral domain B is a function $\deg : B \to \mathbb{N} \cup \{-\infty\}$ with the following properties:

(E1) for $f \in B$, we have $\deg(f) = -\infty$ if and only if $f = 0$;

(E2) for non-zero $f, g \in B$, we have $\deg(f) \leq \deg(fg)$;

(E3) for all $f, g \in B$ with $g \neq 0$, there exist $q, r \in B$ such that $f = gq + r$ and $\deg(r) < \deg(g)$.

The fact that $\mathbb{C}[z]$ is equipped with a Euclidean function implies in particular that it is a principal ideal domain. In conclusion, $\mathbb{C}[z]$ is a principal ideal domain, equipped with an extra valuation ord_∞ on its field of fractions such that the degree formula (3) holds, and such that $\deg = -\mathrm{ord}_\infty$ is a Euclidean function on $\mathbb{C}[z]$.

For the Fargues–Fontaine curve, it turns out that unfortunately the analogue of deg fails to be a Euclidean function, and therefore Fargues and Fontaine introduce the notion of an *almost Euclidean function* $\deg : B \to \mathbb{N} \cup \{-\infty\}$, which is defined by replacing (E3) by the following pair of strictly weaker axioms [16, Déf. 5.2.1]:

(E3′) for $f \in B$ such that $\deg(f) = 0$, we have $f \in B^\times$;

(E3″) for $f, g \in B$ with $\deg(g) \geq 1$, there exist $q, r \in B$ such that $f = gq + r$ and $\deg(r) \leq \deg(g)$.

We reach the following axiomatisation of the algebraic pairs of interest to us (the following simplistic definition does not appear in the work of Fargues and Fontaine, but will be helpful for expository purposes):

Definition 2.4. — An *algebraic* $\mathbb{P}^1$ is a pair (B, ν) consisting of a principal ideal domain B and a valuation $\nu : \mathrm{Frac}(B) \to \mathbb{Z} \cup \{\infty\}$ such that $-\nu$ is an almost Euclidean function on B. We say that the pair is *complete*[(5)] if the degree formula

$$\nu(f) + \sum_{\mathfrak{p} \subseteq B} \mathrm{ord}_{\mathfrak{p}}(f) = 0$$

is satisfied for all $f \in B$. (As in (4), here $\mathfrak{p}$ runs over the non-zero prime ideals of B and $\mathrm{ord}_{\mathfrak{p}}$ denotes the associated $\mathfrak{p}$-adic valuation on B.)

As explained above, the prototypical example of an algebraic $\mathbb{P}^1$ is $(\mathbb{C}[z], \mathrm{ord}_\infty)$.

We may now give an algebraic statement approximating the existence of the Fargues–Fontaine curve in terms of the classical crystalline and de Rham period rings B_{crys} and B_{dR} of p-adic Hodge theory. These are large $\mathbb{Q}_p$-algebras which have been central in p-adic Hodge theory since their introduction by Fontaine [20, 21, 24]; their role in the theory of p-adic Galois representations will be reviewed in §3.3. Their precise definitions (which may be found in §5.3) are for the moment unimportant as we will only need the following three properties: B_{dR} is a complete discrete valuation field (on which we will denote the valuation by ν_{dR}) of residue characteristic zero [21, §1.5.5]; B_{crys} is a subring of B_{dR} [21, §4.1]; B_{crys} is equipped with an endomorphism φ known as its Frobenius [21, §2.3.4]. Let $B_e := B_{\mathrm{crys}}^{\varphi=1}$ be the ring of Frobenius-fixed points of B_{crys}, and continue to write ν_{dR} for the restriction of ν_{dR} to $\mathrm{Frac}(B_e) \subseteq B_{\mathrm{dR}}$.

Theorem 2.5 (Fargues–Fontaine). — *The pair (B_e, ν_{dR}) is a complete algebraic $\mathbb{P}^1$.*

Theorem 2.5 succinctly encodes a range of fundamental results about the period rings. Firstly, the theorem states that the ring B_e is in fact a principal ideal domain; the first indication of this was Berger's surprising result [4, Prop. 1.1.9] that it is a Bézout ring (i.e., all its finitely generated ideals are principal).

Secondly, the fact that $-\nu_{\mathrm{dR}}$ is an almost Euclidean function on B_e turns out to be a formal algebraic consequence (see [16, §5.2]) of the classical *fundamental exact sequence* of the period rings

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \longrightarrow 0 \tag{5}$$

[6, Prop. 1.17] [8, Prop. 1.3(iv)], where $B_{\mathrm{dR}}^+ := \{f \in B_{\mathrm{dR}} : \nu_{\mathrm{dR}}(f) \geq 0\}$ is the ring of integers of the discrete valuation field B_{dR}. Thus the fundamental exact sequence translates into the existence of an almost Euclidean function, which we will see in a moment corresponds to a cohomological vanishing statement on the Fargues–Fontaine curve. Thus we see a first example of how the geometry of the curve encodes algebraic results of p-adic Hodge theory.

The problem with Theorem 2.5 is that it is not a geometric statement: there is no actual curve. To remedy this, we adopt first the following definition (which was already used in Theorem 2.3 when stating the existence of X^{FF}):

(5) By analogy with Riemann surfaces one might prefer to say "compact", but "complete" is the more traditional terminology in algebraic geometry.

Definition 2.6 ([16, Déf. 5.1.1]). — A *curve* is a regular, Noetherian, separated, connected, one-dimensional scheme X; it is *complete* if the degree formula $\sum_{x\in X}\operatorname{ord}_x(f)=0$ holds for all f in the function field of X.[(6)]

Curves are related to algebraic $\mathbb{P}^1$s by elementary algebraic geometry as follows. Suppose that X is a complete curve together with a chosen point $\infty\in X$ such that $X\setminus\{\infty\}$ is affine, say $\operatorname{Spec}(B)$. The point ∞ defines a valuation $\operatorname{ord}_\infty$ on $\operatorname{Frac}(B)$ (= the function field of X), and we wish to characterise whether the pair $(B,\operatorname{ord}_\infty)$ is an algebraic $\mathbb{P}^1$ purely in terms of geometric properties of X and ∞:

– One defines the degree of a Weil divisor $\sum_{x\in X} n_x[x]$ to be $\sum_{x\in X} n_x$; the assumption that X is complete states that this is trivial on the divisor of any rational function, and thus it induces $\deg:\operatorname{Pic}(X)\to\mathbb{Z}$. It is straightforward to see that B is a principal ideal domain if and only if the degree map $\deg:\operatorname{Pic}(X)\to\mathbb{Z}$ is an isomorphism [16, Lem. 5.4.1].

Let $\mathcal{O}_X(1):=\mathcal{O}_X(\infty)$ be the line bundle on X associated to the Weil divisor $[\infty]$, and more generally $\mathcal{O}_X(k)=\mathcal{O}_X(1)^{\otimes k}$ the line bundle associated to $k[\infty]$ for any $k\in\mathbb{Z}$.

– It is easy to check that $-\operatorname{ord}_\infty$ defines a Euclidean function on B if and only if $H^1(X,\mathcal{O}_X(k))=0$ for all $k\geq -1$ [16, Prop. 5.4.2], e.g., $X=\mathbb{P}^1_{\mathbb{C}}$.

– Similarly, $-\operatorname{ord}_\infty$ defines an almost Euclidean function on B if and only if $H^1(X,\mathcal{O}_X(k))=0$ for all $k\geq 0$ [16, Prop. 5.4.2], e.g., $X=X^{\text{FF}}$ by Theorem 2.7.

– The completeness of X tautologically implies the completeness of the pair $(B,\operatorname{ord}_\infty)$.

In conclusion, the geometric and cohomological hypotheses

$$\deg:\operatorname{Pic}(X)\overset{\simeq}{\to}\mathbb{Z}\qquad\text{and}\qquad H^1(X,\mathcal{O}_X(k))=0\text{ for all }k\geq 0 \tag{6}$$

imply that the pair $(B=H^0(X\setminus\{\infty\},\mathcal{O}_X),\operatorname{ord}_\infty)$ is a complete algebraic $\mathbb{P}^1$ in the sense of Definition 2.4.

Fargues and Fontaine prove that the pair (B_e,ν_{dR}) from Theorem 2.5 arises in this way from their curve. To state the result, we should mention that the de Rham and crystalline period rings implicitly depended on having chosen an untilt of F, thereby giving rise to a preferred point $\infty\in X^{\text{FF}}$ via Theorem 2.3.

Theorem 2.7 (Fargues–Fontaine). — *X^{FF} is a complete curve which, together with the point ∞, satisfies properties (6) ; the associated algebraic $\mathbb{P}^1$ is (B_e,ν_{dR}).*

Thus the Fargues–Fontaine curve “glues together” the period rings of p-adic Hodge theory to form a complete curve with similar geometric and cohomological properties to the Riemann sphere. Given the unwieldy nature and enormity of the period rings, it is remarkable that they underlie a reasonable geometric object. We warn the reader

(6) More generally we could also include the data of a specified integer $\deg(x)\geq 1$ for each closed point $x\in X$, and replace the degree formula by $\sum_{x\in X}\deg(x)\operatorname{ord}_x(f)=0$. However, in this survey we have chosen to focus entirely on the Fargues–Fontaine curve associated to an algebraically closed field F, in which case the degrees $\deg(x)$ turn out to all be 1 and hence may be suppressed.

however that, although the ring of globally regular functions on the Fargues–Fontaine curve is $H^0(X^{\mathrm{FF}}, \mathcal{O}_{X^{\mathrm{FF}}}) = \mathbb{Q}_p$ (to prove this, calculate Čech cohomology of $\mathcal{O}_{X^{\mathrm{FF}}}$ using the fundamental exact sequence (5)), the curve is not at all of finite type over $\mathbb{Q}_p$: indeed, Theorem 2.3 tells us that its residue fields are all algebraically closed (hence are not finite extensions of $\mathbb{Q}_p$).

2.3. First definition of the curve

Now we address the following question, which will motivate our first definition of X^{FF}: assuming the validity of Theorem 2.7, how can we reconstruct the curve X^{FF} from the algebraic data (B_e, ν_{dR})?

In the case of the Riemann sphere from §2.2, we may recover $\mathbb{P}^1_{\mathbb{C}}$ from the corresponding pair $(\mathbb{C}[z], \mathrm{ord}_\infty = -\deg)$ by observing that there is an isomorphism of graded rings

$$\mathbb{C}[z_0, z_1] \xrightarrow{\simeq} \bigoplus_{k\geq 0} \mathrm{Fil}_k\, \mathbb{C}[z], \qquad z_0, z_1 \mapsto z, 1 \in \mathrm{Fil}_1\, \mathbb{C}[z],$$

where $\mathrm{Fil}_k\, \mathbb{C}[z] := \{f \in \mathbb{C}[z] : \deg f \leq k\}$. Therefore $\mathbb{P}^1_{\mathbb{C}} \cong \mathrm{Proj}(\bigoplus_{k\geq 0} \mathrm{Fil}_k\, \mathbb{C}[z])$.

Analogously for the Fargues–Fontaine curve, Theorem 2.7 suggests that X^{FF} might be Proj of the graded ring $\bigoplus_{k\geq 0} \mathrm{Fil_k}\, B_e$, where $\mathrm{Fil}_k\, B_e := \{b \in B_e : \nu_{\mathrm{dR}}(b) \geq -k\}$. It is customary to rewrite this isomorphically by using the following classical facts about the crystalline period ring: there exists a $\mathbb{Q}_p$-subalgebra $B^+_{\mathrm{crys}} \subseteq B_{\mathrm{crys}}$ and an element $t \in B^+_{\mathrm{crys}}$ such that

– $B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[\frac{1}{t}]$,
– $\varphi(t) = pt$ and $\nu_{\mathrm{dR}}(t) = 1$, and
– if an element $b \in B_{\mathrm{crys}} \cap B^+_{\mathrm{dR}}$ satisfies $\varphi(b) = p^k b$ for some $k \geq 0$ then $b \in B^+_{\mathrm{crys}}$ [21, Th. 5.3.7(i)]; in short, $B^{\varphi=p^k}_{\mathrm{crys}} \cap B^+_{\mathrm{dR}} = B^{+\,\varphi=p^k}_{\mathrm{crys}}$.

It formally follows that we have a bijection $\mathrm{Fil_k}\, B_e \xrightarrow{\simeq} P^{\mathrm{FF}}_k := B^{+\,\varphi=p^k}_{\mathrm{crys}}$, $b \mapsto bt^k$ for each $k \geq 0$. We may assemble these into an isomorphism of graded rings

$$\bigoplus_{k\geq 0} \mathrm{Fil}_k\, B_e \xrightarrow{\simeq} P^{\mathrm{FF}} = \bigoplus_{k\geq 0} B^{+\,\varphi=p^k}_{\mathrm{crys}}$$

and so reach a definition of the Fargues–Fontaine curve:

Definition 2.8. — The *Fargues–Fontaine curve* is the scheme $X^{\mathrm{FF}} := \mathrm{Proj}(P^{\mathrm{FF}})$.

Revisiting the earlier results, Theorem 2.7 asserts in particular that $\mathrm{Proj}(P^{\mathrm{FF}})$ is a complete curve satisfying (6). This turns out to be a formal consequence of the following central result in the theory:

Theorem 2.9 (Fargues–Fontaine). — *The graded ring P^{FF} is graded factorial with irreducible elements of degree one. That is, any non-zero element $f \in P^{\mathrm{FF}}_k$ may be written uniquely (up to re-ordering and multiples of $\mathbb{Q}_p^\times$) as a product $f = t_1 \cdots t_k$, where $t_i \in P^{\mathrm{FF}}_1$.*

Theorem 2.9 should be compared to the analogous fact that $\mathbb{C}[z_0, z_1]$ is also graded factorial with irreducible elements of degree one: indeed, any homogeneous polynomial in $\mathbb{C}[z_0, z_1]$ may be written uniquely (up to re-ordering and multiples of $\mathbb{C}^\times$) as a product of linear homogeneous polynomials. Moreover, just as points of $\mathbb{P}^1_{\mathbb{C}}$ correspond to $\mathbb{C}$-lines in the two-dimensional vector space $\mathrm{Fil}_1\,\mathbb{C}[z] = \mathbb{C} + \mathbb{C}z$ via

$$\mathbb{P}^1_{\mathbb{C}} \xrightarrow{\simeq} (\mathrm{Fil}_1\,\mathbb{C}[z])/\mathbb{C}^\times, \qquad x \mapsto \begin{cases} z - x & x \in \mathbb{C}, \\ 1 & x = \infty, \end{cases}$$

so too do points of $\mathrm{Proj}(P^{\mathrm{FF}})$ correspond to $\mathbb{Q}_p$-lines in the infinite dimensional $\mathbb{Q}_p$-vector space P^{FF}_1.

In fact, Theorem 2.9 is the essential result underlying the construction of X^{FF}: Fargues and Fontaine check directly, using only elementary algebraic geometry, that if a graded ring $P = \bigoplus_{k \geq 0} P_k$ is graded factorial with irreducibles of degree one and P_0 is a field (and a more technical hypothesis which we omit, but which in the case of the Fargues–Fontaine curve essentially follows from the fundamental exact sequence (5)), then $\mathrm{Proj}(P)$ is a complete curve satisfying (6) for any choice of closed point ∞. See [16, Ths. 5.2.7 & 6.5.2].

However, it seems difficult to directly prove Theorem 2.9 using classical results about the crystalline period ring. Instead, Fargues and Fontaine return to the untilting point of view of §2.1 and reconstruct the graded ring P^{FF} by studying various rings of functions on $|Y|$. Their construction is the focus of §5.

3. VECTOR BUNDLES

In this section we explain how vector bundles on the Fargues–Fontaine curve are related to p-adic Galois representations and to various categories of linear algebraic objects which have appeared in p-adic Hodge theory. We begin by reviewing the classical slope theory of Harder and Narasimhan, which applies to vector bundles on any curve.

3.1. Harder–Narasimhan theory

To review briefly the classical theory of Harder and Narasimhan, let X be a Riemann surface such as $\mathbb{P}^1_{\mathbb{C}}$, or more generally a smooth projective curve over any algebraically closed field. To any vector bundle E on X we may associate two fundamental invariants:

– its *rank* $\mathrm{rk}\,E \in \mathbb{N}$;

– its *degree* $\deg E \in \mathbb{Z}$, defined to be the degree of its determinant line bundle; recall here that the degree of a line bundle L is defined, for example, by identifying L with a Weil divisor $\sum_{x \in X} n_x[x]$ up to rational equivalence and setting $\deg L := \sum_{x \in X} n_x$.

From these one defines the third invariant of E, its *slope* $\mu(E) := \deg E / \mathrm{rk}\,E \in \mathbb{Q}$. The vector bundle E is said to be *semi-stable* if $\mu(E') \leq \mu(E)$ for every sub-bundle

$E' \subseteq E$.[7] The following classical theorem was proved by Harder and Narasimhan in 1974 [25] and is fundamental in the study of vector bundles on curves:

THEOREM 3.1 (Harder–Narasimhan [25]). — *Let E be a vector bundle on X. Then E posseses a unique filtration by sub-bundles*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

with the following two properties:

– *the quotient bundle E_i/E_{i-1} is semi-stable for each $i = 1, \ldots, m$, and*
– $\mu(E_1/E_0) > \cdots > \mu(E_m/E_{m-1})$.

There exist several axiomatisations of Harder and Narasimhan's theory, the most general of which is due to André [1]. We do not adopt André's formalism in greatest generality, but the special case which appears in [10, 16]. Let $\mathcal{E}$ be a category and assume that to each object $E \in \mathcal{E}$ there are associated two invariants

$$\operatorname{rk} E \in \mathbb{N}, \qquad \deg E \in \mathbb{Z}$$

(which only depend on E up to isomorphism), and then define the slope of E as $\mu(E) := \deg E / \operatorname{rk} E$. These are subject to the following additional assumptions in order that the theory should work:

(HN1) $\mathcal{E}$ is an exact category (e.g., projective/free modules over a ring – possibly equipped with extra structure such as a filtration, endomorphism, etc. – or vector bundles on a scheme/space);

(HN2) the rank function rk factors through an abelian category in a manner subject to certain hypotheses [16, §5.5.1], which we do not reproduce here.

Under these assumptions, the category $\mathcal{E}$ behaves like vector bundles on a curve: in particular, any object of $\mathcal{E}$ possesses a Harder–Narasimhan filtration as in Theorem 3.1.

We now turn to examples of categories which are subject to this axiomatic Harder–Narasimham formalism; each category should be viewed as vector bundles on a curve in a generalised sense, even if the curve is not evident.[8]

3.1.1. *Vector bundles on a curve.* — Let X be a Riemann surface, or more generally a smooth projective curve over a field, or even more generally any complete curve in the sense of Definition 2.6; let $\operatorname{Vect}(X)$ be the category of vector bundles on X. Equipped with the usual notion of rank and degree, as defined above, the category $\operatorname{Vect}(X)$ satisfies the necessary axioms and we recover Harder–Narasimhan's original theory.

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(7) By *sub-bundle* we mean that E' is locally a direct summand of E.

(8) Strictly speaking this is slightly misleading: if we impose extra structure such as a polarisation, then the Harder–Narasimham formalism can in fact be extended to higher dimensional varieties [1, §3.1.2].

3.1.2. *Vector bundles on* $\mathbb{P}^1_{\mathbb{C}}$. — We now restrict attention to $\mathbb{P}^1_{\mathbb{C}}$ and give an algebraic description of its vector bundles purely in terms of the pair $(\mathbb{C}[z], \mathrm{ord}_\infty)$ from §2.2.

Let E be a vector bundle on $\mathbb{P}^1_{\mathbb{C}}$. The sections of E on the complex plane form a finite free $\mathbb{C}[z]$-module M, while the completed germ of sections of E at the remaining point $\infty \in \mathbb{P}^1_{\mathbb{C}}$ form a finite free $\mathbb{C}[[z_\infty]]$-module M_∞, where $z_\infty = 1/z$ is our preferred coordinate at ∞. The fact that the modules M and M_∞ arise from the same vector bundle corresponds to an identification of M_∞ as a $\mathbb{C}[[z_\infty]]$-lattice inside the finite dimensional $\mathbb{C}((z_\infty))$-vector space $M \otimes_{\mathbb{C}[z]} \mathbb{C}((z_\infty))$. In this way we arrive at the following equivalence of categories(9) which describes vector bundles on $\mathbb{P}^1_{\mathbb{C}}$ purely in terms of the pair $(\mathbb{C}[z], \mathrm{ord}_\infty)$:

$$\begin{aligned}\mathrm{Vect}(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\simeq} \{&\text{pairs } (M, M_\infty), \text{ where } M \text{ is a finite free } \mathbb{C}[z]\text{-module and} \\ &M_\infty \text{ is a } \mathbb{C}[[z_\infty]]\text{-lattice inside } M \otimes_{\mathbb{C}[z]} \mathbb{C}((z_\infty))\}.\end{aligned}$$

Under this equivalence, the rank of a vector bundle is given by $\mathrm{rk}_{\mathbb{C}[z]} M = \mathrm{rk}_{\mathbb{C}[[z_\infty]]} M_\infty$, while the degree can be expressed by comparing bases of the free modules M and M_∞.

3.1.3. (B, ν_∞)*-pairs : vector bundles on an algebraic* $\mathbb{P}^1$. — Let (B, ν) be a complete algebraic $\mathbb{P}^1$, in the sense of Definition 2.4. Motivated by Example 3.1.2, we may define a (B, ν)*-pair*, or more informally a "vector bundle over the pair (B, ν)", to be the data of a pair (M, M_∞), where M is a finite free B-module and M_∞ is an $\mathcal{O}_\nu$-lattice inside the finite dimensional K_ν-vector space $M \otimes_B K_\nu$. Here K_ν is the completion of $K = \mathrm{Frac}(B)$ with respect to the discrete valuation ν, and $\mathcal{O}_\nu$ denotes its ring of integers. In the case of the pair $(\mathbb{P}^1_{\mathbb{C}}, \mathrm{ord}_\infty)$, these are precisely $\mathbb{C}((z_\infty))$ and $\mathbb{C}[[z_\infty]]$.

The rank of a pair (M, M_∞) is defined to be the rank of the module M (or, equivalently, of M_∞), while its degree is defined by comparing bases of M and M_∞ as in §3.1.2. The category of (B, ν)-pairs is thus subject to the Harder–Narasimhan formalism.

Suppose that (B, ν) arises from a complete curve X and a chosen point $\infty \in X$ satisfying hypotheses (6), as in §2.2. Then the same argument as in §3.1.2 shows that (B, ν)-pairs identify with actual vector bundles on X, and that moreover this identification is compatible with rank and degree.

In the case of the algebraic $\mathbb{P}^1$ of the Fargues–Fontaine curve (B_e, ν_{dR}), such pairs were introduced first by Berger and called more simply *B-pairs* [4, §2],(10) whence our terminology. Repeating the above definition for clarity, a (B_e, ν_{dR})*-pair* is a pair (M, M_{dR}) where M is a finite free B_e-module and M_{dR} is a B^+_{dR}-lattice inside $M \otimes_{B_e} B_{\mathrm{dR}}$. (Here we have implicitly used that the inclusion $\mathrm{Frac}(B_e) \subseteq B_{\mathrm{dR}}$ becomes an equality upon ν_{dR}-adic completion, but this is an easy consequence of the surjectivity in the fundamental exact sequence (5).) By the previous paragraph, Theorem 2.7 implies the following:

(9)Proving that this functor is really an equivalence of categories is a well-known application of the Beauville–Laszlo theorem [2].

(10)More precisely, Berger's B-pairs were also equipped with a Galois action.

Proposition 3.2 ([16, §8.2.1.1]). — *The category of* (B_e, ν_{dR})*-pairs identifies with the category* $\mathrm{Vect}(X^{\mathrm{FF}})$ *of actual vector bundles on the Fargues–Fontaine curve.*

3.1.4. *Filtered vector spaces.* — Given a field extension L/F, let $\mathrm{VectFil}_{L/F}$ be the category of pairs $(V, \mathrm{Fil}^\bullet V_L)$, where V is a finite dimensional F-vector space and $\mathrm{Fil}^\bullet$ is a separated and exhaustive filtration on $V_L := V \otimes_F L$. The rank and degree of $(V, \mathrm{Fil}^\bullet V_L)$ are defined to be

$$\mathrm{rk}(V, \mathrm{Fil}^\bullet V_L) := \dim_F V, \qquad \deg(V, \mathrm{Fil}^\bullet V_L) := \sum_{i \in \mathbb{Z}} i \dim_L(\mathrm{gr}^i V_L).$$

The category $\mathrm{VectFil}_{L/F}$ is thus subject to the Harder–Narasimhan formalism.

3.1.5. *Isocrystals.* — Now we turn to an example from the theory of p-adic cohomology, which will be used to build vector bundles on the Fargues–Fontaine curve. Let k be a perfect field of characteristic p and $K_0 := \mathrm{Frac}(W(k))$, where $W(k)$ denotes the p-typical Witt vectors of k. Recall that K_0 is a complete discrete valuation field of mixed characteristic $(0, p)$ and with residue field k; moreover, the Frobenius automorphism $\varphi : k \xrightarrow{\simeq} k$, $x \mapsto x^p$, induces a field automorphism of K_0, still denoted by φ. For example, if $k = \mathbb{F}_q$, then $K_0 = \mathbb{Q}_p(\zeta_{q-1})$ and $\varphi \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$ is uniquely characterised by the identity $\varphi(\zeta_{q-1}) = \zeta_{q-1}^p$. We will review Witt vectors further at the start of §5.1.

An *isocrystal over* k is a pair (D, φ_D) where D is a finite dimensional K_0-vector space and $\varphi_D : D \to D$ is a φ-semilinear isomorphism (here *φ-semilinear* means $\varphi_D(ad) = \varphi(a)\varphi_D(d)$ for all $a \in K_0$ and $d \in D$). The rank and degree of an isocrystal are defined by

$$\mathrm{rk}(D, \varphi_D) := \dim_{K_0} D, \qquad \deg(D, \varphi_D) := -\deg^+ \det(D, \varphi_D),$$

where $\deg^+$ of a rank one isocrystal (L, φ_L) is defined by choosing any basis element $e \in L$, writing $\varphi_L(e) = ae$ for some $a \in K_0$, and setting $\deg^+(L, \varphi_L) := \nu_p(a)$. The category $\varphi\text{-}\mathrm{Mod}_{K_0}$ of isocrystals is then subject to the Harder–Narasimhan formalism.(11)

For example, given a rational number $\lambda \in \mathbb{Q}$ written uniquely as $\lambda = d/h$ with $d, h \in \mathbb{Z}$, $h > 0$, $(d, h) = 1$, we may define an isocrystal $(D_\lambda, \varphi_\lambda) \in \varphi\text{-}\mathrm{Mod}_{K_0}$ as follows. We set $D_\lambda = K_0^h$ as a K_0-vector space, with basis elements $e_1, \dots, e_h$, and define $\varphi_\lambda : D_\lambda \to D_\lambda$ to be the unique φ-semilinear endomorphism satisfying

$$\varphi_\lambda(e_i) = \begin{cases} e_{i+1} & i = 1, \dots, h-1, \\ p^{-d} e_1 & i = d. \end{cases}$$

The isocrystal $(D_\lambda, \varphi_\lambda)$ has rank h, degree d, and slope λ.

(11)The seemingly strange minus sign in front of $\deg^+$ is designed to ensure that Lemma 3.3 and the resulting functor $\mathcal{E}(-)$ are compatible with degrees. In fact, $\deg^+$ itself is also a valid degree function on $\varphi\text{-}\mathrm{Mod}_{K_0}$, leading to a different Harder–Narasimhan formalism. The two associated Harder–Narasimhan filtrations are the same but run in opposite directions, i.e., they are split; this gives the classical Dieudonné–Manin decomposition of the isocrystal.

3.2. Classification of vector bundles on the Fargues–Fontaine curve

We now focus our attention to vector bundles on the Fargues–Fontaine curve X^{FF}; recall from Proposition 3.2 that these correspond to (B_e, ν_{dR})-pairs. Berger observed that these may be constructed from isocrystals over $\mathbb{F}_p$ as follows [4, Ex. 2.1.2]; we omit the proof, although it is not difficult:

LEMMA 3.3. — *Let $(D, \varphi_D) \in \varphi\text{-}\mathrm{Mod}_{\mathbb{Q}_p}$ be an isocrystal over $\mathbb{F}_p$. Then*

$$((B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} D)^{\varphi=1}, B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} D)$$

is a (B_e, ν_{dR})-pair, with rank and degree (as in §3.1.3) given by the rank and degree of the isocrystal (D, φ_D) (as in §3.1.5)

The lemma functorially associates to any isocrystal (D, φ_D) a vector bundle on X^{FF}, which is denoted by $\mathcal{E}(D, \varphi_D)$. In the particular case of the isocrystal $(D_\lambda, \varphi_\lambda)$, for $\lambda \in \mathbb{Q}$, we write instead $\mathcal{O}_{X^{\mathrm{FF}}}(\lambda) := \mathcal{E}(D_\lambda, \varphi_\lambda)$. Since both Proposition 3.2 and Lemma 3.3 are compatible with rank and degree, we see that the vector bundle $\mathcal{O}_{X^{\mathrm{FF}}}(\lambda)$ has rank h, degree d, and slope λ.

The following is the classification theorem of vector bundles on X^{FF}:

THEOREM 3.4 (Fargues–Fontaine [16, Thm. 8.2.10]). — *Let E be a vector bundle on X^{FF}. Then there exists a unique sequence of rational numbers $\lambda_1 \geq \cdots \geq \lambda_m$ such that E is isomorphic to $\bigoplus_{i=1}^m \mathcal{O}_{X^{\mathrm{FF}}}(\lambda_i)$*

COROLLARY 3.5. — *(1) The functor $\mathcal{E}(-) : \varphi\text{-}\mathrm{Mod}_{\mathbb{Q}_p} \to \mathrm{Vect}(X^{\mathrm{FF}})$ is essentially surjective.*

(2) Let E be a vector bundle on X and let $\lambda \in \mathbb{Q}$. Then E is semi-stable of slope λ if and only if it is isomorphic to $\mathcal{O}_{X^{\mathrm{FF}}}(\lambda)^m$ for some $m \geq 1$.

(3) The category of semi-stable, slope-zero vector bundles on X^{FF} is equivalent to the category of finite dimensional $\mathbb{Q}_p$-vector spaces, via

$$E \mapsto H^0(X^{\mathrm{FF}}, E), \qquad V \mapsto V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X^{\mathrm{FF}}}.$$

Proof. — The proof of the classification theorem is a tour de force beyond the scope of this text, requiring p-divisible groups and Hodge–Tate period mappings. We refer to [16, §8] for details and to [17, §6.3] for a sketch.

The corollary is an easy consequence of the theorem. (For part (3), the reader should recall that $H^0(X^{\mathrm{FF}}, \mathcal{O}_{X^{\mathrm{FF}}}) = \mathbb{Q}_p$, as we saw just after Theorem 2.7.) □

Remark 3.6. — Theorem 3.4 may be used to show that the Fargues–Fontaine curve is geometrically simply connected [16, Th. 8.6.1]. In other words, every finite étale cover of X^{FF} is of the form $X^{\mathrm{FF}} \otimes_{\mathbb{Q}_p} E$ for some finite extension E of $\mathbb{Q}_p$, and hence the étale fundamental group of the curve is the absolute Galois group of $\mathbb{Q}_p$:

$$\pi_1^{\text{ét}}(X^{\mathrm{FF}}) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p).$$

Since the goal of p-adic Hodge theory, local class field theory, and the local Langlands programme is, loosely, to understand $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, the above identification epitomises the way in which the curve offers a new geometric approach to these subjects.

3.3. An application to p-adic Galois representations

We are now ready to present one of the main applications of the theory of vector bundles on the Fargues–Fontaine curve to the theory of p-adic Galois representations, which will reveal the strength of the classification theorem.

Let K be an extension of $\mathbb{Q}_p$ and $G_K = \mathrm{Gal}(\overline{K}/K)$ its absolute Galois group; let $K_0 = \mathrm{Frac}(W(k)) \subseteq K$ be the maximal unramified subextension of K, where k is the residue field of K. The reader is welcome to assume $K = K_0 = \mathbb{Q}_p$. The study of p-adic Galois representations is concerned with continuous representations of G_K on finite dimensional $\mathbb{Q}_p$-vector spaces; we denote the category of such representations by $\mathrm{Rep}(G_K)$. Fontaine classically defined various classes of such representations, notably Hodge–Tate, de Rham, crystalline and (potentially) semi-stable representations in terms of his period rings, which reflect different types of reduction on the geometric side of the picture. We now recall his formalism [21, 22]. Let B be a (typically large) $\mathbb{Q}_p$-algebra, equipped with an action by G_K; write $F = B^{G_K}$ for its subalgebra of fixed elements, which we assume is a field. Given a p-adic Galois representation $V \in \mathrm{Rep}(G_K)$, we may consider the F-vector space $D_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_K}$, where G_K acts diagonally on $B \otimes_{\mathbb{Q}_p} V$. The representation V is said to be *B-admissible* if $\dim_F D_B(V) = \dim_{\mathbb{Q}_p} V$, which encodes a certain compatibility between the ring B and the representation V, and one writes $\mathrm{Rep}_B(G_K) \subseteq \mathrm{Rep}(G_K)$ for the set of B-admissible representations. Under certain axiomatic hypotheses on B (namely that it is *G_K-regular* [22, §1.4]), Fontaine showed that the functor

$$D_B : \mathrm{Rep}_B(G_K) \longrightarrow \{\text{finite dim. } F\text{-vector spaces}\}$$

is faithful [22, Prop. 1.5.2] (and moreover exact and compatible with tensor products).

If the algebra B is equipped with additional structure, such as a grading, filtration, endomorphism, etc. compatible with the G_K-action, then this structure is formally inherited by the vector space $D_B(V)$; for example, if φ is an endomorphism of B which is $\mathbb{Q}_p$- and G_K-linear, then $\varphi \otimes 1$ defines an endomorphism of $B \otimes_{\mathbb{Q}_p} V$ and of $D_B(V)$. In that case D_B becomes a functor

$$D_B : \mathrm{Rep}_B(G_K) \longrightarrow \{\text{finite dim. } F\text{-vector spaces with additional structure}\}$$

which we can hope is now fully faithful (since adding extra structure to vector spaces restricts the permitted morphisms). If so, and if we can identify the image of the functor, then we will have a purely linear algebraic description of the category of B-admissible p-adic Galois representations; this is a major goal of p-adic Hodge theory.

Example 3.7. — (1) A p-adic Galois representation is said to be *de Rham* if it is B_{dR}-admissible. Since B_{dR} is a complete discrete valuation field, it is filtered by its valuation

$\mathrm{Fil}^k B_{\mathrm{dR}} := \{b \in B_{\mathrm{dR}} : \nu_{\mathrm{dR}}(b) \geq k\}$. Moreover $B_{\mathrm{dR}}^{G_K} = K$ [21, §1.5.7] and so Fontaine's formalism provides a faithful functor

$$D_{\mathrm{dR}} : \mathrm{Rep}_{\mathrm{dR}}(G_K) \longrightarrow \{\text{finite dim. filtered } K\text{-vector spaces}\}, \qquad V \mapsto (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

(2) A p-adic Galois representation is said to be *crystalline* if it is B_{crys}-admissible. Since B_{crys} is equipped with the Frobenius endomorphism φ and is known to satisfy $B_{\mathrm{crys}}^{G_K} = K_0$ [22, Prop. 5.1.2], Fontaine's formalism yields a faithful functor

$$D_{\mathrm{crys}} : \mathrm{Rep}_{\mathrm{crys}}(G_K) \longrightarrow \varphi\text{-}\mathrm{Mod}_{K_0}, \qquad V \mapsto (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Geometrically, de Rham (resp. crystalline) p-adic Galois representations arise from the étale cohomology of smooth p-adic varieties (resp. of good reduction).

In these examples, although we have embedded de Rham and crystalline representations into linear algebraic categories, neither are the embeddings fully faithful nor have we identified the images; therefore we cannot claim to have yet provided a linear algebraic description of either class of p-adic Galois representations. Fontaine overcame this problem as follows by simultaneously taking both the filtration and Frobenius into account.

Let $\varphi\text{-}\mathrm{ModFil}_{K/K_0}$ denote the category of triples $(D, \varphi_D, \mathrm{Fil}^\bullet D_K)$ where D is a finite dimensional K_0-vector space, $\varphi_D : D \xrightarrow{\simeq} D$ is a φ-linear isomorphism, and $\mathrm{Fil}^\bullet$ is a separated and exhaustive filtration on $D_K := D \otimes_{K_0} K$; that is, $(D, \varphi_D) \in \varphi\text{-}\mathrm{Mod}_{K_0}$ and $(D, \mathrm{Fil}^\bullet D_K) \in \mathrm{VectFil}_{K/K_0}$. We define rank as usual and a degree by adding those on $\varphi\text{-}\mathrm{Mod}_{K_0}$ and $\mathrm{VectFil}_{K/K_0}$:

$$\begin{aligned} \mathrm{rk}(D, \varphi_D, \mathrm{Fil}^\bullet D_K) &:= \dim_{K_0} D \\ \deg(D, \varphi_D, \mathrm{Fil}^\bullet D_K) &:= \deg(D, \varphi_D) + \deg(D, \mathrm{Fil}^\bullet D_K). \end{aligned}$$

The resulting Harder–Narasimhan formalism on the category of filtered isocrystals $\varphi\text{-}\mathrm{ModFil}_{K/K_0}$ encodes information about the interaction of the Frobenius and the filtration.

If V is a crystalline Galois representation, then Fontaine's formalism (and the inclusion of the period rings $B_{\mathrm{crys}} \subseteq B_{\mathrm{dR}}$) implies that V is also de Rham and that $D_{\mathrm{dR}}(V) = D_{\mathrm{crys}}(V) \otimes_{K_0} K$ [22, §5.1.7]. Therefore combining the two parts of Example 3.7 upgrades D_{crys} to a functor

$$D_{\mathrm{crys}} : \mathrm{Rep}_{\mathrm{crys}}(G_K) \longrightarrow \varphi\text{-}\mathrm{ModFil}_{K/K_0}.$$

This can be shown moreover to be fully faithful by writing down an explicit left inverse

$$\begin{aligned} &V_{\mathrm{crys}} : \varphi\text{-}\mathrm{ModFil}_{K/K_0} \longrightarrow \mathrm{Rep}_{\mathrm{crys}}(G_K) \\ &\qquad (D, \varphi_D, \mathrm{Fil}^\bullet D_K) \mapsto \{v \in B_{\mathrm{crys}} \otimes_{K_0} D : \varphi(v) = v \text{ and } v \in \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K)\} \end{aligned}$$

(this will gain a natural geometric interpretation in terms of the Fargues–Fontaine curve in the proof of Theorem 3.8).

Fontaine conjectured that a filtered isocrystal was in the essential image of D_{crys} if and only if it was *weakly admissible* [19, §5.2.6] [22, Conj. 5.4.4]; we do not reproduce Fontaine's original definition of weakly admissible, but remark that in terms of the Harder–Narasimhan formalism on φ-ModFil_{K/K_0} defined above, it is tautologically equivalent to asking that it be semi-stable and of slope zero. The "only if" direction in Fontaine's conjecture is easy [22, Prop. 5.4.2]. Note that the resolution of the conjecture gives an equivalence of categories

$$D_{\mathrm{crys}} : \mathrm{Rep}_{\mathrm{crys}}(G_K) \xrightarrow{\simeq} \varphi\text{-}\mathrm{ModFil}^{\mathrm{w.a.}}_{K/K_0},$$

where w.a. denotes the full subcategory of weakly admissible filtered isocrystals. This indeed resolves a goal of p-adic Hodge theory by describing a large class of p-adic Galois representations purely in terms of linear algebraic data.

Fontaine's conjecture was proved first by Colmez–Fontaine in 2000 [8] and a different proof was given by Berger in 2008 [4, 5]. Both these papers contained important ideas which contributed to the discovery of the Fargues–Fontaine curve. Conversely, the Classification Theorem 3.4 of vector bundles on X^{FF} provides a short conceptual proof of Fontaine's conjecture:

Theorem 3.8 (Colmez–Fontaine). — *Fontaine's above conjecture is true.*

Proof. — In §3.2 we associated to any isocrystal $(D, \varphi_D) \in \varphi$-$\mathrm{Mod}_{\mathbb{Q}_p}$ a vector bundle $\mathcal{E}(D, \varphi_D)$ on X^{FF}. More generally, given a filtered isocrystal $D = (D, \varphi_D, \mathrm{Fil}^\bullet D_K) \in \varphi$-$\mathrm{ModFil}_{K/K_0}$, it may be checked that

$$((B_{\mathrm{crys}} \otimes_{K_0} D)^{\varphi=1}, \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K))$$

is a (B_e, ν_{dR})-pair and hence defines a vector bundle $\mathcal{E}(D, \varphi_D, \mathrm{Fil}^\bullet D_K)$ on X^{FF}, of the same rank and degree as D.

Now suppose that D is semi-stable of slope zero, i.e., weakly admissible, and put $E := \mathcal{E}(D, \varphi_D, \mathrm{Fil}^\bullet D_K)$. Note that

$$H^0(X^{\mathrm{FF}}, E) = (B_{\mathrm{crys}} \otimes_{K_0} D)^{\varphi=1} \cap \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K) = V_{\mathrm{crys}}(D),$$

where the first equality is a Čech cohomology calculation on the curve, and the second equality is by definition. This is the cohomological interpretation of $V_{\mathrm{crys}}(D)$ which we mentioned earlier.

Since D is semi-stable of slope zero, the vector bundle E is also: this is not completely automatic from Harder–Narasimhan formalism, but is not hard to check [7, Prop. 5.6(iii)]. Therefore Corollary 3.5 states that E is constant in the sense that

$$H^0(X^{\mathrm{FF}}, E) \otimes_{\mathbb{Q}_p} \mathcal{O}_{X^{\mathrm{FF}}} \xrightarrow{\simeq} E,$$

whence $\dim_{\mathbb{Q}_p} V_{\mathrm{crys}}(D) = \dim_{K_0} D$. This equality of dimensions forces D to coincide with $D_{\mathrm{crys}}(V_{\mathrm{crys}}(D))$ [8, Prop. 4.5], which completes the proof. □

Remark 3.9. — More generally the above argument describes the category of semi-stable[(12)] p-adic Galois representations as weakly admissible filtered isocrystals with monodromy operator. This was also conjectured by Fontaine [21, §5.4.4] and proved first by Colmez–Fontaine. We refer the reader to Colmez's preface [7, §5.2].

4. DIAMONDS, D'APRÈS SCHOLZE

We now change direction slightly to give an introduction to Scholze's theory of diamonds [31, 32] and the relation to the Fargues–Fontaine curve. This will require overviewing some aspects of the theory of adic and perfectoid spaces.

4.1. Huber pairs

The theory of adic spaces, developed by Huber [26], is an approach to rigid analytic geometry, i.e., a theory of algebraic geometry in which topologies such as the p-adic topology are taken into account. The class of topological rings for which the theory works are the Huber rings: a topological ring R is called *Huber* if there exists an open subring $R_0 \subseteq R$ and a finitely generated ideal I of R_0 such that the subspace topology on R_0 is the I-adic topology. The possibilities $R_0 = R$ or $I = 0$ are not excluded. We stress that R_0 and I are not part of the data: in general there are many choices. The easiest way to build a Huber ring is to start with a ring R_0 and a non-zero-divisor $\pi \in R_0$, and to give $R := R_0[\frac{1}{\pi}]$ the topology with basis $f + \pi^m R_0$, for $f \in R$, $m \geq 0$; such a Huber ring R is called a *Tate ring.* For example, given a field K topologised by a non-archimedean absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$, the rings K and $K\langle T\rangle := \{\sum_{n\geq 0} a_n T^n : a_n \in K,\ a_n \to 0 \text{ as } n \to \infty\}$ are Tate rings.

Huber enriches the topology by specifying which elements of R should correspond to bounded functions: this enrichment takes the form of a chosen subring $R^+ \subseteq R$ which is open, integrally closed in R, and such that each element $f \in R^+$ is "power-bounded" (i.e., the sequence f^n does not tend to infinity as $n \to \infty$). The pair $\underline{R} = (R, R^+)$ is called a *Huber pair*, or a *Tate–Huber pair* if R is Tate. It is sometimes the case that $R^+ = R_0$, but other times this is not even allowed (R_0 might not be integrally closed); on the other hand there is always a largest possibility for R^+, namely the subring of all power-bounded elements of R. In the case of the field K of the previous paragraph, one almost always takes $K^+ = \mathcal{O}_K$, and so writes K as shorthand for the Tate–Huber pair $(K, \mathcal{O}_K)$.

We will be mainly interested in Tate–Huber pairs;[(13)] these are, in short, the "rings" which will underly our geometric objects (just as usual rings underlie schemes). A

[(12)] This use of "semi-stable" is completely unrelated to the semi-stable appearing in the Harder–Narasimhan theory.

[(13)] For experts: we ignore the fact that the pair might not be sheafy.

homomorphism of Tate–Huber pairs is a continuous map which restricts to the $+$-subrings.

4.2. Perfectoid pairs

We say that a Tate–Huber pair $\underline{S} = (S, S^+)$ is *perfect* if S is topologically complete and is a perfect $\mathbb{F}_p$-algebra (i.e., each element of S has a unique p^{th}-root).

Dropping the characteristic p assumption, a Tate–Huber pair $\underline{T} = (T, T^+)$ is said to be *perfectoid* [23, 30] if T is topologically complete and there exists an element $\pi \in T^+$ such that

– the topology on T^+ is the π-adic topology;
– $p \in \pi^p T^+$;
– given $f \in T^+$ there exists $g \in T^+$ such that $f \equiv g^p \bmod \pi T^+$.

The hypotheses imply that T^+ (and hence T) has many p-power roots. A Tate–Huber pair of characteristic p is perfectoid if and only if it is perfect [31, Prop. 3.5].

For example, if C (resp. F) is a field as in (Pf_0) (resp. (Pf_p)), then $(C, \mathcal{O}_C)$ (resp. $(F, \mathcal{O}_F)$) is perfectoid. Moreover, the tilting formalism we have explained for fields in §2.1 extends to perfectoid pairs: given a perfectoid Tate–Huber pair $\underline{T} = (T, T^+)$, its *tilt* $\underline{T}^\flat = (T^\flat, T^{+\flat})$ is a perfect Tate–Huber pair. Here the rings $T^\flat$ and $T^{+\flat}$ are defined exactly as we did for C in line (1), namely by equipping the set of compatible p-power sequences in T and T^+ with termwise multiplication and addition as in line (2).

4.3. Properties and examples of diamonds

Just as any ring A gives rise to a scheme by taking its spectrum $\mathrm{Spec}(A)$ of prime ideals, any Tate–Huber pair $\underline{R}$ gives rise to a diamond spectrum $\mathrm{Spd}(\underline{R})$. Although we will not define the category of diamonds until §4.4, we may nevertheless state some properties of this process (at the risk of oversimplifying the theory).

Firstly, we warn the reader that the contravariant functor $\underline{R} \mapsto \mathrm{Spd}(\underline{R})$ loses information, unless we impose a suitable smoothness or perfectoid hypothesis: in particular, we will see that the contravariant functor

$$\mathrm{Spd} : \text{Aff-Perf} := \{\text{perfect Tate–Huber pairs}\} \longrightarrow \{\text{diamonds}\} \tag{7}$$

is fully faithful.

Given a diamond X, its *points* with value in a perfect Tate–Huber pair $\underline{S}$ are the elements of the set

$$X(\underline{S}) := \mathrm{Hom}_{\text{diamonds}}(\mathrm{Spd}(\underline{S}), X).$$

Such points provide a conservative family of test objects: a morphism $f : X \to Y$ of diamonds is an isomorphism if and only if $X(\underline{S}) \to Y(\underline{S})$ is a bijection for all perfect Tate–Huber $\underline{S}$. In particular, any diamond X is determined by its associated functor of points

$$X(-) : \text{Aff-Perf} \longrightarrow \text{Sets},$$

and so we may use this perspective to completely describe some diamonds.

For example, the functor of points associated to the diamond $\mathrm{Spd}(\mathbb{Q}_p) := \mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p)$ will turn out to be

$$\underline{S} \mapsto \{\text{equivalence classes of untilts of } \underline{S}\}. \tag{8}$$

Similarly to §2.1 for fields, here an *untilt* of $\underline{S}$ is a pair $(\underline{T}, \iota)$ consisting of a perfectoid Tate–Huber pair $\underline{T}$ of characteristic zero (i.e., $T \supseteq \mathbb{Q}_p$) and an isomorphism $\iota : \underline{S} \overset{\simeq}{\to} \underline{T}^\flat$;[14] the equivalence relation is as for fields. Since the category of diamonds has products we may form a new diamond $\mathrm{Spd}(\mathbb{Q}_p) \times \mathrm{Spd}(\mathbb{Q}_p)$, which was implicitly mentioned in the introduction: its $\underline{S}$-points are pairs of untilts of $\underline{S}$.

More generally, given any Tate–Huber pair $\underline{R} = (R, R^+)$, we may look at *untilts over $\underline{R}$ of $\underline{S}$*: by definition such an untilt is a triple $(\underline{T}, \iota, f)$ consisting of a perfectoid Tate–Huber pair $\underline{T}$, an isomorphism $\iota : \underline{S} \overset{\simeq}{\to} \underline{T}^\flat$, and a homomorphism $f : \underline{R} \to \underline{T}$. The notion of equivalence is the obvious one. The functor of points associated to $\mathrm{Spd}(\underline{R})$ is precisely

$$\underline{S} \mapsto \{\text{equivalence classes of untilts over } \underline{R} \text{ of } \underline{S}\}.$$

Using this we can check the following: given any perfectoid Tate–Huber $\underline{T}$, then there is a natural identification of diamonds

$$\mathrm{Spd}(\underline{T}) = \mathrm{Spd}(\underline{T}^\flat).$$

Indeed, in light of the above description of these diamonds, this follows from the following tilting equivalence:

Proposition 4.1 (Scholze [30, Th. 5.2 & Lem. 6.2]). — *Let $\underline{T}$ be a perfectoid Tate–Huber pair. Then tilting induces an equivalence of categories*

$$\{\text{perfectoid Tate–Huber pairs over } \underline{T}\} \overset{\simeq}{\to} \{\text{perfect Tate–Huber pairs over } \underline{T}^\flat\}$$

Proof. — We stress that this is one of the easier tilting equivalences in the theory: it is essentially proved by repeating the arguments of Proposition 5.1 after replacing $W(\mathcal{O}_{C^\flat})$ by $W(T^+)$. See [30] [31, §3] for details. □

Any algebraic variety X over $\mathbb{Q}_p$ gives rise to a diamond $X^\diamond$. Indeed, X is built from the spectra of various $\mathbb{Q}_p$-algebras, from which one may build Tate–Huber pairs via a p-adic completion process, then glue the associated diamond spectra. The process $X \mapsto X^\diamond$ should be viewed as a generalisation of tilting, which now applies to finite-type objects such as varieties, whereas tilting itself only applies to (large) perfectoid objects. Moreover, this process does not lose any information under suitable hypotheses:[15]

(14) As in §2.1, we abusively abbreviate "untilt of characteristic zero" to "untilt".

(15) For experts, let us explain what is really true. Firstly, we may replace $\mathbb{Q}_p$ by any complete extension K. Then any rigid analytic variety X over K admits a diamond-ification $X^\diamond \in \mathrm{Diam}$, and the functor

$$\{\text{seminormal rigid analytic varieties over } K\} \longrightarrow \{\text{diamonds over } \mathrm{Spd}\, K\}, \qquad X \mapsto X^\diamond$$

Proposition 4.2 (Scholze [31, Prop. 10.2.4] + GAGA). — *The above diamondification functor*

$$\{\text{proper smooth varieties over } \mathbb{Q}_p\} \longrightarrow \{\text{diamonds over } \operatorname{Spd}\mathbb{Q}_p\}, \qquad X \mapsto X^\diamond$$

is fully faithful.

Remark 4.3. — The theory of diamonds also contains geometric objects which do not arise from varieties. One of the most important is the B_{dR}^+*-affine Grassmannian.* We restrict to the case of GL_n, although the following discussion continues to work for other algebraic groups.

It is known classically that the quotient

$$\mathrm{GL}_n(\mathbb{C}((t)))/\,\mathrm{GL}_n(\mathbb{C}[[t]]),$$

which classifies $\mathbb{C}[[t]]$-lattices inside $\mathbb{C}((t))^n$, admits the structure of an ind complex analytic space; alternatively, from an algebraic point of view, the functor

$$\mathbb{C}\text{-Algs} \ni A \mapsto \mathrm{GL}_n(A((t)))/\,\mathrm{GL}_n(A[[t]]) = \{A[[t]]\text{-lattices inside } A((t))\} \tag{9}$$

is represented by an ind projective scheme. This geometric object is known as the affine Grassmannian and is fundamental in the geometric Langlands programme [3, 28] (notably in the geometric Satake correspondence).

In p-adic arithmetic geometry the analogues of $\mathbb{C}((t))$ and $\mathbb{C}[[t]]$ are B_{dR} and B_{dR}^+, as we have seen in §2.2, and therefore one considers instead the quotient $\mathrm{GL}_n(B_{\mathrm{dR}})/\,\mathrm{GL}_n(B_{\mathrm{dR}}^+)$, which classifies B_{dR}^+-lattices inside B_{dR}^n. Recall that B_{dR} and B_{dR}^+ implicitly depend on both F and a chosen untilt (C,ι). More generally, given any perfect Tate–Huber pair $\underline{S}$ and chosen untilt over C (i.e., morphism $\operatorname{Spd}(\underline{S}) \to \operatorname{Spd}(C)$), it is possible to define analogues of these de Rham period rings and therefore an analogue of (9) on the category of perfect Tate–Huber pairs over $\operatorname{Spd}(C)$. This so-called B_{dR}^+*-affine Grassmannian* is an ind diamond [32, §19] which is expected to appear in the arithmetic Satake correspondence.

4.4. Definition of diamonds via adic spaces

As we mentioned in §4.1, Huber pairs are the building blocks of Huber's approach to rigid analytic geometry. More precisely, to each Huber pair $\underline{R} = (R, R^+)$ one associates its *adic spectrum* $\operatorname{Spa}(\underline{R})$, which is a topological space of continuous absolute values $|\cdot|$ on R such that $|f| \leq 1$ for all $f \in R^+$. Such valuations arise from choosing a prime ideal $\mathfrak{p} \subseteq R$ and writing down an absolute value on the fraction field of $R/\mathfrak{p}$; since the trivial absolute value is allowed, we see that $\operatorname{Spa}(\underline{R})$ is a refinement of $\operatorname{Spec}(R)$ by taking into account the topology on R and the boundedness of R^+.

Just as varieties and schemes are built by gluing together the spectra of rings, adic spaces are built by gluing together the adic spectra of Huber pairs. As for schemes,

is fully faithful. The seminormality hypothesis is a necessary consequence of the fact that all perfectoid rings are seminormal and so cannot detect the difference between a ring and its seminormalisation. To obtain the proposition we restrict to the proper smooth case and apply GAGA.

there is a robust theory of étale cohomology for adic spaces, due to Huber under suitable finiteness hypotheses [26] and to Scholze in greater generality [31].

A *perfectoid space* is an adic space built by gluing the adic spectra of perfectoid Tate–Huber pairs. It can be shown (though it is somewhat technical [31, §6]) that the tilting process $\underline{T} \mapsto \underline{T}^\flat$ is compatible with gluing; therefore one may associate to any perfectoid space Z its tilt $Z^\flat$, which is a perfectoid space of characteristic p (i.e., an adic space built by gluing the adic spectra of perfect Tate–Huber pairs).

Since perfectoid spaces are rather large, it is better to replace the étale topology by a *pro-étale* variant [31, §8], in which infinite limits of étale covers are allowed. Scholze uses this pro-étale topology on perfectoid spaces to define diamonds as follows:

Definition 4.4 (Scholze [31, Def. 11.1] [32, §8]). — Let Perf denote the site of perfectoid spaces of characteristic p equipped with the pro-étale topology. A *diamond* X is a sheaf of sets on Perf of the form $X = Z/\mathcal{R}$, where $Z, \mathcal{R} \in$ Perf and $\mathcal{R} \hookrightarrow Z \times Z$ is a equivalence relation such that the two projection morphisms $\mathcal{R} \to Z$ are pro-étale. (Here we identify Z with the sheaf $\mathrm{Hom}_{\mathrm{Perf}}(-, Z)$ on Perf, and similarly for $\mathcal{R}$.)

In short, a diamond X is an algebraic space for the site Perf. Informally, X is obtained by gluing perfectoid spaces of characteristic p along pro-étale overlaps.

To relate this definition to the discussion in §4.3, the key is to check the following, in which we globalise in the obvious way the notion of "untilts over":

Proposition 4.5 (Scholze [31, Prop. 15.4]). — *Let $\underline{R}$ be a Tate–Huber pair. Then*

$$\mathrm{Spd}(\underline{R}) : \mathrm{Perf} \longrightarrow \mathrm{Sets}, \qquad Z \mapsto \{\textit{equivalence classes of untilts over } \mathrm{Spa}(\underline{R}) \textit{ of } Z\}$$

is a diamond in the sense of Definition 4.4.

Proof. — By adding to R many p-power roots of elements it is possible to construct a perfectoid-isation $\underline{R}_\infty$ of $\underline{R}$, such that the resulting map of adic spectra $\mathrm{Spa}(\underline{R}_\infty) \to \mathrm{Spa}(\underline{R})$ is a pro-étale cover [31, Prop. 15.4].

Proposition 4.1 then shows that $\mathrm{Spd}(\underline{R})$, as defined in the statement of the current proposition, is a quotient of the representable sheaf $\mathrm{Hom}_{\mathrm{Perf}}(-, \mathrm{Spa}(\underline{R}_\infty^\flat))$. It remains to check that we are quotienting by a pro-étale equivalence relation, which Scholze does by establishing various general results about pro-étale torsors. □

More generally, Proposition 4.5 associates a diamond $X^\diamond$ to any analytic adic space X, given by

$$X^\diamond : \mathrm{Perf} \longrightarrow \mathrm{Sets}, \qquad Z \mapsto \{\text{equivalence classes of untilts over } X \text{ of } Z\}.$$

Here "analytic" means that that X is built from Tate–Huber pairs, not general Huber pairs, so Proposition 4.5 may be glued to define $X^\diamond$. As already mentioned in §4.3, the diamond $X^\diamond$ should be viewed as a generalised tilt of X.

We now invite the reader to return to §4.3 with the definition of diamonds and check some of the claims we made there. For example, the fully faithful embedding of line (7) is simply Spa : Aff-Perf $\hookrightarrow$ Perf followed by the Yoneda embedding, while the functor of points view is valid since diamonds are by definition sheaves on Perf.

In [31], Scholze puts diamonds on a firm geometric and cohomological footing by establishing a theory of étale cohomology, including a six functor formalism. The theory of diamonds is part of the framework in which Fargues and Scholze carry out their work on the local Langlands correspondence [18, 32].

4.5. The Fargues–Fontaine curve as a diamond

We now return to the Fargues–Fontaine curve $X^{\mathrm{FF}} = X^{\mathrm{FF}}_F$ associated to the field F as in (Pf_p). The appearance of topologies, convergent series, etc. in the construction of X^{FF} from §5 is a reflection of the fact that there exists a closely related adic space $\mathcal{X}^{\mathrm{FF}}$, known as the *adic Fargues–Fontaine curve* [12]. To construct it, let $\mathcal{Y}$ be the adic space obtained by removing the vanishing loci of the elements $p, [\pi] \in A_{\mathrm{inf}}$ from $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$. Here A_{inf} is defined in §5.1, π is an arbitrary non-zero element of $\mathfrak{m}_F$, and the topology on A_{inf} is the $(p, [\pi])$-adic topology. Just as we will see for the topological space $|Y|$ in §5.2, the Frobenius on A_{inf} induces a totally discontinuous Frobenius action on $\mathcal{Y}$, and so $\mathcal{X}^{\mathrm{FF}} := \mathcal{Y}/\varphi^{\mathbb{Z}}$ is a well-defined adic space. Thus the adic Fargues–Fontaine curve may be defined relatively quickly.

There is a morphism of ringed spaces $\mathcal{X}^{\mathrm{FF}} \to X^{\mathrm{FF}}$ under which $\mathcal{X}^{\mathrm{FF}}$ behaves like an analytification of the scheme X^{FF}. In particular, a GAGA theorem asserts that the two spaces have the same vector bundles and cohomology [27, Th. 8.7.7] [12, Th. 3.5].

As explained in §4.4, there is then a diamond $\mathcal{X}^{\mathrm{FF}\diamond}$ associated to the adic Fargues–Fontaine curve. This turns out to have a beautifully simple description in terms of the diamond spectra of our fields:

Theorem 4.6 (Scholze [31, §15.2.6]). — *There are natural isomorphisms of diamonds*

$$\mathcal{Y}^{\diamond} \cong \mathrm{Spd}(F) \times \mathrm{Spd}(\mathbb{Q}_p), \qquad \mathcal{X}^{\mathrm{FF}\diamond} \cong \mathrm{Spd}(F)/\varphi^{\mathbb{Z}} \times \mathrm{Spd}(\mathbb{Q}_p).$$

Proof. — We will sketch the first isomorphism, the second then being obtained by modding out by the Frobenius action on F.

According to the functor of points perspective explained in §4.3, we must show that $\mathcal{Y}^{\diamond}$ and $\mathrm{Spd}(F) \times \mathrm{Spd}(\mathbb{Q}_p)$ naturally have the same $\underline{S}$-points for all perfect Tate–Huber pairs $\underline{S}$. We calculate these points:

– By the full faithfullness of (7), an $\underline{S}$-point of $\mathrm{Spd}(F)$ is simply a morphism of Tate–Huber pairs $f : (F, \mathcal{O}_F) \to (S, S^+)$.

– By (8), an $\underline{S}$-point of $\mathrm{Spd}(\mathbb{Q}_p)$ is an untilt $(\underline{T}, \iota)$ of $\underline{S}$.

– Using the definitions of $\mathcal{Y}_F$ and the diamond-ification process, it can be shown that an $\underline{S}$-point of $\mathcal{Y}^{\diamond}$ is an untilt $(\underline{T}, \iota)$ of $\underline{S}$ together with a continuous homomorphism $A_{\mathrm{inf}} \to T^+$ which extends to $A_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] \to T$ (see the proof of [32, Prop. 11.2.1]).

Therefore, to produce a map of diamonds $\mathrm{Spd}(F) \times \mathrm{Spd}(\mathbb{Q}_p) \to \mathcal{Y}^\diamond$ we should show, given any morphism $f : (F, \mathcal{O}_F) \to (S, S^+)$ and untilt $(\underline{T}, \iota)$ of $\underline{S}$, how to produce a natural continuous homomorphism $A_{\mathrm{inf}} \to T^+$ which extends to $A_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] \to T$. For this homomorphism we take

$$A_{\mathrm{inf}} = W(\mathcal{O}_F) \longrightarrow W(S^+) \cong W(T^{+\flat}) \xrightarrow{\theta_T} T^+,$$

where the first arrow is induced by f, the isomorphism is induced by ι, and the θ map is a generalisation of Fontaine's map from Proposition 5.1. (Indeed, this whole construction is essentially the same argument as Proposition 5.1, but now phrased in the diamond language.)

Conversely, every such homomorphism $A_{\mathrm{inf}} \to T^+$ can be constructed in this way (by a universal property of Witt vectors), which means that the map of diamonds is an isomorphism. $\square$

Throughout this text we focus on the Fargues–Fontaine curve associated to a field F with properties (Pf_p); in fact, Fargues and Fontaine work more generally with the condition "F is perfect" rather than "F is algebraically closed". Using perfectoid spaces and diamonds much more possible: there exists a Fargues–Fontaine "relative curve over Z" $\mathcal{X}_Z^{\mathrm{FF}} = \mathcal{Y}_Z/\varphi^{\mathbb{Z}}$ associated to any perfectoid space Z of characteristic p. The objects $\mathcal{Y}_Z$ and $\mathcal{X}_Z^{\mathrm{FF}}$ are again adic spaces, whose associated diamonds satisfy the analogous formulae of the previous proposition [32, §11.2 & §15.2].

5. CONSTRUCTION OF THE CURVE

We now turn to the actual construction of the Fargues–Fontaine curve and to sketching the main theorems stated in §2. As already mentioned, Fargues and Fontaine do not do this in terms of the classical period rings of §2.2–2.3, but instead adopt the point of view of §2.1 that the curve should be an enrichment of a space of untilts; they therefore introduce and study various rings of "holomorphic functions in p" on the spaces of untilts $|Y|$ and $|Y|/\varphi^{\mathbb{Z}}$.

We have fixed a field F as in (Pf_p) and drop the subscript F from our notation wherever possible (including from the forthcoming rings A_{inf}, B^{b+}, B^b, B^+, B, all of which depend on F).

5.1. The infinitesimal period ring A_{inf}

We begin by recalling some theory of Witt vectors [33, §6]. Let $W(F)$ be the ring of p-typical Witt vectors of the field F: each element of $W(F)$ may be written uniquely as a sequence $(\alpha_0, \alpha_1, \dots)$, where $\alpha_i \in F$, and addition and multiplication are given by certain universal polynomials such as

$$(\alpha_0, \alpha_1, \alpha_2, \dots) + (\beta_0, \beta_1, \beta_2, \dots) = (\alpha_0 + \beta_0, \alpha_1 + \beta_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \alpha_0^i \beta_0^{p-i}, \dots)$$

$$(\alpha_0, \alpha_1, \alpha_2, \dots) \cdot (\beta_0, \beta_1, \beta_2, \dots) = (\alpha_0\beta_0, \alpha_0^p\beta_0 + \beta_0^p\alpha_1 + p\alpha_1\beta_1, \dots)$$

The fact that F is a perfect field of characteristic p means that its Witt vectors are well-understood: $W(F)$ is a complete discrete valuation ring such that $W(F)/pW(F) = F$. Any element of $W(F)$ may be expressed uniquely as a p-adically convergent sum $\sum_{n\geq 0}[\alpha_n]p^n = (\alpha_0, \alpha_1^p, \alpha_2^{p^2}, \dots)$ for some coefficients $\alpha_n \in F$; here $[\alpha] := (\alpha, 0, 0, 0, \cdots)$ denotes the *Teichmüller lift* of an element $\alpha \in F$.

We will in fact be more interested in the Witt vectors $A_{\mathrm{inf}} := W(\mathcal{O}_F)$ of the ring of integers $\mathcal{O}_F$; this is the subring $A_{\mathrm{inf}} \subseteq W(F)$ consisting of series $\sum_{n\geq 0}[\alpha_n]p^n$ such that all the coefficients α_n belong to $\mathcal{O}_F$. The ring A_{inf} is known as the *infinitesimal period ring*, sometimes denoted by $\mathbb{A}$ or $\mathbb{A}_{\mathrm{inf}}$, and may be characterised as the unique p-complete and p-torsion-free ring such that $A_{\mathrm{inf}}/pA_{\mathrm{inf}} = \mathcal{O}_F$ [33, §II.5–II.6]. It is a foundational building block of Fontaine's various p-adic period rings and of the Fargues–Fontaine curve. Although A_{inf} is a "large" ring (in particular, non-Noetherian), it is sufficiently explicit to be amenable.

An element $\xi = \sum_{n\geq 0}[\alpha_n]p^n \in A_{\mathrm{inf}}$ is said to be *primitive* if $\alpha_0 \neq 0$ and there exists $k \geq 0$ such that $\alpha_k \in \mathcal{O}_F^\times$; the smallest such k is called the *degree* of ξ. Let $\mathrm{Prim}_k \subseteq A_{\mathrm{inf}}$ denote the set of primitive elements of degree k. It is easy to see that $\mathrm{Prim}_0 = A_{\mathrm{inf}}^\times$ and that $\mathrm{Prim}_k \cdot \mathrm{Prim}_l \subseteq \mathrm{Prim}_{k+l}$; in particular, if a principal ideal $I \subseteq A_{\mathrm{inf}}$ can be generated by a primitive element of degree k, then any generator of I is a primitive element of degree k. Primitive elements of degree one play an important role thanks to the following correspondence, which identifies the set $|Y|$ as part of the prime ideal spectrum of A_{inf} and is a provisional form of Theorem 2.3.

Proposition 5.1 ([27, Th. 3.6.5] [23, Prop. 1.1.1]). — *There is a natural bijective correspondence between $|Y|$ (i.e., equivalence classes of untilts of F) and the set of ideals of A_{inf} generated by a primitive element of degree one.*

Proof. — Let us explain here how the correspondence is defined and give some notation which will be needed in the rest of the section. See [16, §2.2] for further details.

Given a field C satisfying (Pf_0), there is a distinguished surjective homomorphism $\theta_C : W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C$ which plays an important role throughout p-adic Hodge theory. It was introduced by Fontaine as

$$\theta_C : W(\mathcal{O}_{C^\flat}) \longrightarrow \mathcal{O}_C, \qquad \sum_{n\geq 0}[\alpha_n]p^n \mapsto \sum_{n\geq 0}\alpha_n^\# p^n$$

where $\alpha_n \in \mathcal{O}_{C^\flat}$ and $\#$ is the untilting map from §2.1. The difficulty is to verify that θ_C is a ring homomorphism, which is done by examining the explicit rules for addition and multiplication in Witt vectors, or by judicious use of universal properties. The kernel $\mathrm{Ker}\,\theta_C$ is a principal ideal of $W(\mathcal{O}_{C^\flat})$ generated by $p - [p^\flat]$ (where $p^\flat \in C^\flat$ is the element from Example 2.1), which is manifestly primitive of degree one in $W(\mathcal{O}_{C^\flat})$.

Changing the point of view, now suppose that $y = (C_y, \iota_y)$ is an untilt of F, i.e., a point of $|Y|$. Then the previous paragraph yields a surjective homomorphism

$$\theta_y : A_{\mathrm{inf}} = W(\mathcal{O}_F) \cong W(\mathcal{O}_{C^\flat}) \xrightarrow{\theta_C} \mathcal{O}_C$$

(where the isomorphism is induced by $\iota_y : F \xrightarrow{\sim} C^\flat$) with kernel $\mathfrak{p}_y := \operatorname{Ker}\theta_y$ generated by a primitive element of degree one; such a generator is often denoted by ξ_y, which the previous paragraph shows may be taken to be $p - [\iota_y^{-1}(p^\flat)]$. This defines the desired correspondence

$$|Y| \longrightarrow \{\text{ideals generated by a primitive element of degree one}\}, \qquad y \mapsto \mathfrak{p}_y = \operatorname{Ker}\theta_y.$$

To prove that this is a bijection, one checks directly that if $\mathfrak{p} \subseteq A_{\inf}$ is an ideal generated by a primitive element of degree one, then $(A_{\inf}/\mathfrak{p})[\frac{1}{p}]$ is indeed a field as in (Pf$_0$) which untilts F. $\square$

The proposition has two immediate consequences. Firstly, it means that elements of $A_{\inf}$ define functions on $|Y|$, with varying field of value in the usual sense of algebraic geometry: namely, given $f \in A_{\inf}$ and $y \in |Y|$, we may write $f(y) := \theta_y(f) \in \mathcal{O}_{C_y}$. Strangely, p should be viewed as a variable.

Secondly, Fargues and Fontaine use the proposition to equip the set $|Y|$ with a topological structure [16, §2.3]. Given ideals $I, J \subseteq A_{\inf}$, define the distance $d(I, J)$ between them to be the smallest $\rho \in [0, 1]$ such that $I + \mathfrak{a}_\rho = J + \mathfrak{a}_\rho$, where $\mathfrak{a}_\rho := \{f \in A_{\inf} : |f \bmod p|_F \leq \rho\} \subseteq A_{\inf}$. Restricted to ideals generated by a primitive element of degree one, it is easy to check that $d(-,-)$ is an ultrametric distance function and so turns $|Y|$ into an ultrametric space. Moreover, the space $|Y|$ resembles a punctured open disk in the following sense: given $y \in |Y|$ with corresponding ideal $\mathfrak{p}_y$, define its *distance to the origin* to be $\mathfrak{r}(y) := d(\mathfrak{p}_y, 0)$. This defines a continuous map $\mathfrak{r} : |Y| \longrightarrow (0, 1)$ such that

$$|Y_\rho| := \{y \in |Y| : \mathfrak{r}(y) \geq \rho\}$$

is complete (under the above distance function d) for any $\rho > 0$.

Exploiting this geometric perspective on $|Y|$, Fargues and Fontaine establish the following preliminary form of Theorem 2.9:

Theorem 5.2 (Fargues–Fontaine [16, Th. 2.4.1]). — *Let $f \in A_{\inf}$ be a primitive element of degree $k \geq 1$. Then there exist primitive elements $\xi_1, \ldots, \xi_k \in A_{\inf}$ of degree one such that $f = \xi_1 \cdots \xi_k$.*

Corollary 5.3 (Weierstrass factorisation for $A_{\inf}$). — *Let $f \in A_{\inf}$ be a primitive element of degree $k \geq 1$. Then there exist non-zero $x_1, \ldots, x_k \in \mathfrak{m}_F$ and a unit $u \in A_{\inf}^\times$ such that*

$$f = u(p - [x_1]) \cdots (p - [x_k]).$$

Proofs. — The corollary is obtained from the theorem by recalling that any primitive element of degree one can be written up to unit as $p - [x]$ for some non-zero $x \in \mathfrak{m}_F$ (as we saw in the proof of Proposition 5.1)

To prove the theorem it suffices to show that f, viewed as a function on $|Y|$, has a zero. Indeed, given $y \in |Y|$ such that $f(y) = 0$, this means precisely that f is divisible by any generator ξ_y of $\mathfrak{p}_y$; writing $f = g\xi_y$ with $g \in A_{\inf}$, one sees that g is necessarily primitive of degree $k - 1$ and then proceeds by induction on k.

It remains to show that f has a zero, which is a key result in the theory. Write $f = \sum_{n\geq 0}[\alpha_n]p^n$ and let $\mathrm{Newt}(f)$ denote the decreasing Newton polygon in $\mathbb{R}^2$ associated to the points $(n, -\log_p|\alpha_n|_F)_{n\geq 0}$. If f can be written as in the corollary, then the valuations of the elements x_i are precisely the slopes of $\mathrm{Newt}(f)$. Conversely, Fargues and Fontaine use the slopes of $\mathrm{Newt}(f)$ to find approximate zeros, eventually taking the limit in $|Y|$ (using that each $|Y_\rho|$ is complete) to construct an actual zero of f. □

The appearance of Newton polygons in the previous sketch is not isolated: it is the main technique which Fargues and Fontaine use to analyse elements of A_{inf} and of the forthcoming rings B^{b+}, B^b, B^+, B.

5.2. Holomorphic functions on $|Y|$ in the variable p

As we have just seen in §5.1, it is helpful to view A_{inf} as functions on the topological space $|Y|$. Fargues and Fontaine substantially develop this point of view by introducing further rings of functions B^{b+}, B^b, B^+, B on $|Y|$ as follows. These will fit together as in the following diagram, in which we have also included the period rings for convenience:

$$\begin{array}{ccccccccc}
A_{\mathrm{inf}} & \subseteq & B^{b+} & \subseteq & B^+ = \text{“bounded functions on } |Y|\text{”} & \subseteq & B^+_{\mathrm{crys}} & \subseteq & B^+_{\mathrm{dR}} \\
\cap\vert & & \cap\vert & & \cap\vert & & \cap\vert & & \cap\vert \\
W(F)[\frac{1}{p}] & \supseteq & B^b & \subseteq & B = \text{“functions on } |Y|\text{”} & & B_{\mathrm{crys}} & \subseteq & B_{\mathrm{dR}} \\
 & & & & & & \cup\vert & & \\
 & & & & & & B_e & &
\end{array}$$

We begin by defining the incomplete rings of functions B^{b+} and B^b:

$$B^{b+} := A_{\mathrm{inf}}[\tfrac{1}{p}] = \left\{ \sum_{n\gg-\infty} [\alpha_n]p^n : \alpha_n \in \mathcal{O}_F \right\}$$

$$\cap\vert$$

$$B^b := A_{\mathrm{inf}}[\tfrac{1}{p}, \tfrac{1}{[\pi]}] = \left\{ \sum_{n\gg-\infty} [\alpha_n]p^n : \alpha_n \in F,\ |\alpha_n|_F \text{ is bounded as } n\to\infty \right\}$$

(where π is an arbitrary non-zero element of $\mathfrak{m}_F$). For any $y \in |Y|$, it is easy to check that the homomorphism $\theta_y : A_{\mathrm{inf}} \to \mathcal{O}_{C_y}$ extends to $\theta_y : B^b \to C_y$ (but not all the way to $W(F)[\frac{1}{p}]$), and therefore we may think of B^{b+} and B^b as rings as function on $|Y|$ via $f(y) := \theta_y(f)$.

With this in mind we introduce the Gauss norms

$$||\cdot||_\rho : B^b \to \mathbb{R}_{\geq 0}, \qquad ||\sum_{n\gg-\infty}[\alpha_n]p^n||_\rho := \sup_n |\alpha_n|\rho^n$$

for all $\rho \in (0,1)$. These norms are multiplicative (this is rather subtle and is a novel observation of Fargues and Fontaine [16, Prop. 1.4.9]) and satisfy the maximum modulus principle that $||\cdot||_\rho \leq \max\{||\cdot||_{\rho_1}, ||\cdot||_{\rho_2}\}$ whenever $0 < \rho_1 \leq \rho \leq \rho_2 < 1$. Let B^+ and B be the Fréchet $\mathbb{Q}_p$-algebras obtained by completing B^{b+} and B^b with respect to this family of norms. Given $y \in |Y|$, the homomorphism $\theta_y : B^b \to C_y$ is continuous with respect to the Gauss norm $|| \cdot ||_{\mathfrak{r}(y)}$ and therefore again extends to a continuous homomorphism $\theta_y : B \to C_y$; this shows that B^+ and B may also be viewed as rings of functions on $|Y|$, and also that $\mathfrak{p}_y B$ is a closed maximal ideal of B equal to $\{f \in B : f(y) = 0\}$.

In fact B is the largest reasonable ring of continuous functions on $|Y|$, in the sense that its spectrum of closed maximal ideals identifies with $|Y|$ (this also provides an intrinsic definition of $|Y|$ without reference to primitive elements):

Proposition 5.4 (Fargues–Fontaine [16, Cor. 2.5.4]). — *The association $y \mapsto \mathfrak{p}_y B$ defines a bijection between $|Y|$ and the set of closed maximal ideals of B.*

Proof. — This results from a finer statement about rings B_I: here I is a compact interval of $(0, 1)$ and B_I is defined to be the completion of B^b with respect to the Gauss norms $|| \cdot ||_\rho$ for $\rho \in I$. If $I = [\rho_1, \rho_2]$ then the maximum modulus principle shows that it is sufficient to complete with respect to $\max\{|| \cdot ||_{\rho_1}, || \cdot ||_{\rho_2}\}$, and thus B_I is a $\mathbb{Q}_p$-Banach algebra. Note that, by definition, we have $B = \varprojlim_I B_I$, where I varies over all compact intervals of $(0, 1)$.

The same argument as immediately before the proposition shows that $\mathfrak{p}_y B_I$ is a maximal ideal of B_I for each $y \in |Y|$ such that $\mathfrak{r}(y) \in I$. Fargues and Fontaine show that this association identifies the "closed annulus" $|Y_I| := \{y \in |Y| : \mathfrak{r}(y) \in I\}$ with the set of maximal ideals of B_I, and that moreover B_I is a principal ideal domain. To do this, they write a typical element $f \in B_I$ as a limit $f = \lim_{n\to\infty} f_n$ of elements $f_n \in B^b$ and then argue similarly to the proof of Theorem 5.2: namely, an examination of the Newton polygons of the f_n allows them to construct a factor of f of the form $p - [x]$ for some $x \in \mathfrak{m}_F$ such that $|x|_F \in I$. Since the Newton polygon of f itself has only finitely many slopes (this is the crucial application of I being compact), this process may be repeated finitely many times to factor f into a product of primitive elements of degree one, which proves all the claims about B_I.

The assertion about B itself then follows by taking the limit over all compact intervals $I \subseteq (0, 1)$. □

The previous proof also shows that $|Y|$ is a curve in a certain sense : it is the increasing union of the subspaces $|Y_I|$, each of which identifies with the maximal ideal spectrum of a principal ideal domain B_I. The fact that B_I is a principal ideal domain also implies that its non-zero ideals, as a monoid under multiplication, is isomorphic to $\mathrm{Div}^+(Y_I)$; here $\mathrm{Div}^+(Y_I)$ denotes the monoid of formal finite sums $\sum_{y\in|Y_I|} n_y[y]$ with $n_y \in \mathbb{N}$. Taking the limit over all compact intervals yields a description of the monoid of the non-zero closed ideals of B. To state it, Fargues and Fontaine introduce $\mathrm{Div}^+(Y) = \varprojlim_I \mathrm{Div}(Y_I)$; in other words, $\mathrm{Div}^+(Y)$ is the monoid of formal sums $\sum_{y\in|Y|} n_y[y]$ with $n_y \in \mathbb{N}$ satisfying the following finiteness condition: for any compact interval $I \subseteq (0, 1)$, the support $\{y \in |Y_I| : n_y \neq 0\}$ is finite. Taking the limit obtains:

Corollary 5.5. — $\mathrm{Div}^+(Y)$ *is isomorphic to the monoid of non-zero closed ideals of* B *under multiplication.*

To explicitly write the isomorphism of the previous corollary, let $\mathrm{ord}_y : B \to \mathbb{N} \cup \{\infty\}$ be the discrete valuation associated to any $y \in |Y|$: indeed, we know that $\mathfrak{p}_y B$ is a maximal ideal of B generated by a single element (namely the primitive element of degree one generating $\mathfrak{p}_y$), whence $B_{\mathfrak{p}_y B}$ is a discrete valuation ring and so ord_y is well-defined. The correspondence of the previous corollary is then given by

$$\begin{aligned} \mathrm{Div}^+(Y) &\xrightarrow{\simeq} \{\text{closed ideals of } B\} \\ \sum_{y \in |Y|} n_y[y] &\mapsto \{f \in B : \mathrm{ord}_y(f) \geq n_y \text{ for all } y \in |Y|\}. \end{aligned}$$

Alternatively, note that each non-zero $f \in B$ defines a divisor

$$\mathrm{div}(f) := \sum_{y \in |Y|} \mathrm{ord}_y(f)[y] \in \mathrm{Div}^+(Y),$$

and then the previous correspondence may be rewritten $D \mapsto \{f \in B : \mathrm{div}(f) \geq D\}$.

At this stage we have a good understanding of $|Y|$ and of its ring of functions B. To construct the Fargues–Fontaine curve we must pass to $|Y|/\varphi^{\mathbb{Z}}$ by taking the Frobenius φ into account. Recall from §2.1 that φ acts on $|Y|$ by sending an untilt (C, ι) to $(C, \iota \circ \varphi)$, where the latter φ denotes the absolute Frobenius automorphism of F. On the algebraic side A_{inf} also possesses a Frobenius automorphism, given by $\varphi(\sum_{n \geq 0}[\alpha_n]p^n) := \sum_{n \geq 0}[\alpha_n^p]p^n$; this is easily seen to extend to automorphisms of B^{b+}, B^b, B^+ and B, and to induce isomorphisms $\varphi : B_{[\rho_1, \rho_2]} \xrightarrow{\simeq} B_{[\rho_1^p, \rho_2^p]}$ [16, §1.6.1].

An element $\xi \in A_{\mathrm{inf}}$ is primitive of degree one if and only if $\varphi^{-1}(\xi)$ is, thereby inducing an automorphism φ of the set of ideals generated by a primitive element of degree one, which is compatible with the correspondence of Proposition 5.1. The action of φ on $|Y|$ satisfies $d(\varphi(y_1), \varphi(y_2)) = d(y_1, y_2)^{1/p}$ and $\mathfrak{r}(\varphi(y)) = \mathfrak{r}(y)^{1/p}$; in particular, the action of the group $\varphi^{\mathbb{Z}}$ on the topological space $|Y|$ is properly discontinuous, whence the quotient $|Y|/\varphi^{\mathbb{Z}}$ inherits the structure of a Hausdorff space for which the quotient map $\pi : |Y| \to |Y|/\varphi^{\mathbb{Z}}$ is a local homeomorphism.

This also shows that $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$, the monoid of formal finite sums of points of $|Y|/\varphi^{\mathbb{Z}}$, identifies with $\{D \in \mathrm{Div}^+(Y) : \varphi^* D = D\}$ via the pull-back

$$\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}}) \hookrightarrow \mathrm{Div}^+(Y), \qquad \sum_{y \in |Y|/\varphi^{\mathbb{Z}}} n_y[y] \mapsto \sum_{y \in |Y|} n_{\pi(y)}[y] \tag{10}$$

[16, Lem. 6.2.3]. Using this identification, one clearly has $\mathrm{div}(f) \in \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$ for any non-zero $f \in B^{\varphi = p^k}$, thereby giving rise to a morphism of monoids

$$\mathrm{div} : \bigsqcup_{k \geq 0} (B \setminus \{0\})^{\varphi = p^k} \longrightarrow \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}}).$$

Fargues and Fontaine establish the following fundamental isomorphism relating such Frobenius eigenspaces of B to the space $|Y|/\varphi^{\mathbb{Z}}$; from this we will easily construct the curve X^{FF} and deduce its main properties in §5.3.

Theorem 5.6 (Fargues–Fontaine [16, Th. 6.2.1]). — *The morphism of monoids*

$$\mathrm{div} : (\bigsqcup_{k\geq 0}(B\setminus\{0\})^{\varphi=p^k})/\mathbb{Q}_p^\times \longrightarrow \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$$

is an isomorphism.

Proof. — We first prove injectivity. Given non-zero $f, g\in B$ with $\mathrm{div}(f)=\mathrm{div}(g)\in\mathrm{Div}^+(Y)$, projecting to $\mathrm{Div}^+(Y_I)$ (which we have seen is isomorphic to the monoid of non-zero ideals of the principal ideal domain B_I) shows that the images of f, g in B_I differ by a (unique) unit; this being true for every compact interval I, taking the limit implies there is $u\in B^\times$ such that $f=gu$. If moreover $f\in B^{\varphi=p^k}$ and $g\in B^{\varphi=p^{k'}}$ then $u\in B^{\varphi=p^{k-k'}}$. But Newton polygon arguments show that

$$B^{\varphi=p^{k-k'}}=\begin{cases}\mathbb{Q}_p & k'=k\\ 0 & k'>k\end{cases}$$

[16, Prop. 4.1.1, Prop. 4.1.2].

Next we explain the proof of surjectivity. Identifying $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$ with a submonoid of $\mathrm{Div}^+(Y)$ as in (10), we see that $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$ is generated by its elements of the form $\sum_{n\in\mathbb{Z}}[\varphi^n(y)]$, for $y\in|Y|$. Therefore it is enough, given any $y\in|Y|$, to find $t_y\in B^{\varphi=p}$ satisfying $\mathrm{div}(t_y)=\sum_{n\in\mathbb{Z}}[\varphi^n(y)]$, i.e., a function t_y with a simple zero at each $\varphi^{\mathbb{Z}}$-translate of y and no other zeros or poles. Let $\xi_y=p-[x]\in A_{\mathrm{inf}}$ be a primitive element of degree one corresponding to y, where $x\in\mathfrak{m}_F$.

Consider first the infinite product

$$\Pi^+(\xi_y):=\prod_{n\geq 0}\varphi^n(\tfrac{\xi_y}{p})=\prod_{n\geq 0}(1-\tfrac{[x^{p^n}]}{p}).$$

This product converges in B^+ and satisfies $\xi_y\varphi(\Pi^+(\xi_y))=p\Pi^+(\xi_y)$ and $\mathrm{div}(\Pi^+(\xi_y))=\sum_{n\geq 0}[\varphi^n(y)]$. Secondly, the existence of Artin–Schreier roots and $p-1^{\mathrm{st}}$ roots in F implies that, given any $g\in B^b$, the equation $\varphi(T)=gT$ has a non-zero solution in B^b [16, Prop. 6.2.0]; in particular there exists a non-zero element $\Pi^-(\xi_y)\in B^b$ satisfying $\varphi(\Pi^-(\xi_y))=\xi_y\Pi^-(\xi_y)$, which can be checked to satisfy automatically $\mathrm{div}(\Pi^-(\xi_y))=\sum_{n<0}[\varphi(y)]$ [16, Lem. 6.2.12]. In conclusion, the element $t_y:=\Pi^-(\xi_y)\Pi^+(\xi_y)\in B^{\varphi=p}$ satisfies $\mathrm{div}(t_y)=\sum_{n\in\mathbb{Z}}[\varphi^n(y)]$, as desired. □

5.3. Second definition of the curve and proofs of the main theorems

We now explain how the results of the section thus far imply the main theorems we have stated about the Fargues–Fontaine curve, namely Theorems 2.3, 2.7, 2.9. We begin with the definition of the curve (which we will soon see does not conflict with Definition 2.8).

Definition 5.7 ([16, Déf. 6.5.1]). — The *Fargues–Fontaine curve* is the scheme

$$X^{\mathrm{FF}}:=\mathrm{Proj}(\bigoplus_{k\geq 0}B^{\varphi=p^k}).$$

We have the following fundamental consequence of Theorem 5.6:

Corollary 5.8 ([16, Th. 6.2.1]). — *The graded ring* $\bigoplus_{k\geq 0} B^{\varphi=p^k}$ *is graded factorial with irreducible elements of degree* 1.

Proofs. — Using a Newton polygon argument once again, one checks that if $f \in B^{\varphi=p^k}$ then $\operatorname{div}(f)$ has degree k [16, Prop. 6.2.6] (here the degree of a divisor $\sum_{y\in|Y|/\varphi^{\mathbb{Z}}} n_y[y]$ is defined as usual to be $\sum_{y\in|Y|/\varphi^{\mathbb{Z}}} n_y$). Therefore the morphism of monoids of Theorem 5.6 is even an isomorphism of graded monoids. But $\operatorname{Div}^+(Y/\varphi^{\mathbb{Z}})$ is by definition the free monoid on its degree one elements, whence the theorem tells us that the same is true of $(\bigsqcup_{k\geq 0}(B\setminus\{0\})^{\varphi=p^k})/\mathbb{Q}_p^\times$, which is precisely the desired unique factorisation claim. □

As explained in the paragraph after Theorem 2.9, the previous unique factorisation corollary more-or-less formally implies that $X^{\text{FF}} = \operatorname{Proj}(\bigoplus_{k\geq 0} B^{\varphi=p^k})$ is a complete curve in the sense of Definition 2.6. Let us now briefly indicate why it satisfies all the theorems we have stated about it.

5.3.1. *Relation of X^{FF} to untilts.* — Theorem 2.3, namely that the points of X^{FF} correspond to $|Y|/\varphi^{\mathbb{Z}}$, is now easy to check. Indeed, Corollary 5.8 implies that the points of X^{FF} correspond to $(B\setminus\{0\})^{\varphi=p}$ up to units; but in turn this corresponds to $|Y|/\varphi^{\mathbb{Z}}$ by the graded isomorphism of Theorem 5.6.

5.3.2. *Relation of X^{FF} to classical period rings.* — Let us now explain how X^{FF} is related to the classical de Rham and crystalline period rings of §2.2–2.3. As mentioned before Theorem 2.7, these period rings depend on the choice of an untilt (C,ι) of F; to simplify notation we suppress ι and write $F = C^\flat$. Let $\infty \in |Y|$ be the corresponding point, and also write ∞ for the resulting points of $|Y|/\varphi^{\mathbb{Z}}$ and X^{FF}. Let $\xi = \xi_\infty \in A_{\text{inf}} = W(\mathcal{O}_{C^\flat})$ be a primitive element of degree one generating the ideal $\operatorname{Ker}\theta_C$ corresponding to ∞.

The period ring B^+_{crys} from §2.3 is by definition obtained by adjoining to A_{inf} the "divided power" elements $\xi^n/n!$ for all $n\geq 1$, then p-completing, then inverting p:

$$B^+_{\text{crys}} := \widehat{A_{\text{inf}}[\tfrac{\xi^n}{n!} : n\geq 1]}[\tfrac{1}{p}].$$

The Frobenius automorphism φ of A_{inf} extends to an endomorphism φ of B^+_{crys}, and elementary manipulations with Gauss norms show that there is a natural identification $B^+ = \bigcap_{n\geq 0}\varphi^n(B^+_{\text{crys}})$ [16, §1.10.3], or in other words B^+ is the largest subalgebra of B^+_{crys} on which φ is an automorphism. We then have equalities of eigenspaces

$$B^{+\,\varphi=p^k}_{\text{crys}} = B^{+\varphi=p^k} = B^{\varphi=p^k}, \tag{11}$$

the first equality being a trivial consequence of the previous sentence, and the second following from a Newton polygon argument [16, Prop. 4.1.3]. Therefore Definitions 2.8 and 5.7 coincide, and so in particular Corollary 5.8 is exactly Theorem 2.9.

We now sketch part of the proof of Theorem 2.7, namely that the algebraic $\mathbb{P}^1$ associated to X^{FF} really is (B_e, ν_{dR}). Let $t = t_\infty \in B^{\varphi=p}$ be the element constructed in the course of the proof of Theorem 5.6; this is the element t alluded to in §2.3, and in particular $B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[\frac{1}{t}]$. By definition of the Proj construction and the fact that t was constructed to vanish only on the set $\varphi^{\mathbb{Z}}(\infty) \subseteq |Y|$, it follows that $X^{\mathrm{FF}} \setminus \{\infty\}$ is Spec of the ring $\bigcup_{k\geq 1} B^{+\,\varphi=p^k} t^{-k}$, which may be re-written using (11) as $B^{\varphi=1}_{\mathrm{crys}} = B_e$, as desired.

Next, the ring of integers B^+_{dR} of the de Rham period ring is by definition the ξ-adic completion of $A_{\mathrm{inf}}[\frac{1}{p}]$. But we saw at the beginning of §5.2 that the map $\theta_C : A_{\mathrm{inf}}[\frac{1}{p}] \to C$ extended to $\theta_C : B \to C$, and therefore it is equivalent to take the ξ-adic completion of B. But the composition $B^{\varphi=p} \hookrightarrow B \xrightarrow{\theta_C} C$ can be shown to be surjective by observing that it identifies with the logarithm of a Lubin–Tate group, whence an elementary argument about local rings of the Proj construction shows that B^+_{dR} is also the completed local ring of the point $\infty \in X^{\mathrm{FF}}$. Related arguments reprove the fundamental exact sequence (5). See [16, §6.4–6.5] for further details.

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